TÉCNICO
LISBOA

Introduction to Mathematical Finance

Exercises

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Contents

Notation, useful results and abbreviations

$W(t)$	Denote standard a Brownian motion by $\{W(t) : t \geq 0\}$. We may also write W_t instead of $W(t)$.
A^c	Given a set A, we denote the complement of A (elements which are not in A) by A^c.
$\mathbb{E}[X]$	Given a probability space $(\Omega, \mathcal{F}, P)$, we denote the expectation with respect to P by $\mathbb{E}[\,\cdot\,]$, i.e., for a random variable $X : \Omega \to \mathbb{R}$, $\mathbb{E}[X] = \int_\Omega X(\omega)\, dP(\omega)$. If X is a random variable with a probability density function of $f(x)$, then the expected value is defined as $\mathbb{E}[X] = \int_\mathbb{R} x f(x)\, dx$.
$\mathrm{Cov}(X, Y)$	Given two random variables X and Y defined on a probability space $(\Omega, \mathcal{F}, P)$, we denote the covariance by $\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$. In particular, if $X = Y$, we denote the variance of X by $\mathrm{Var}(X)$.
$\mathcal{N}(\sigma, \mu^2)$	For $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, we denote the normal distribution with mean σ and variance σ^2 by $\mathcal{N}(\sigma, \mu^2)$.
$X \sim D$	If a random variable X has distribution D, we write $X \sim D$.
$X \overset{d}{=} Y$	If two random variables X and Y have the same distribution, we write $X \overset{d}{=} Y$.
$X \in \mathcal{L}^1$	A random variable X defined on $(\Omega, \mathcal{F},\mathrm{P})$ is integrable, which we denote by $X \in \mathcal{L}^1$, if $\mathbb{E}[\lvert X \rvert] < \infty$.
$\mathbb{E}[X \vert \mathcal{G}]$	Given a random variable $X \in \mathcal{L}^1$ defined on $(\Omega, \mathcal{F}, P)$ and $\mathcal{H}$ a sub-σ-algebra of $\mathcal{F}$, we denote the conditional expectation of X with respect to $\mathcal{G}$ by $\mathbb{E}[X \vert \mathcal{H}]$.
$[X, Y]_t$	Given two (stochastic) processes $X = \{X_t : t \geq 0\}$ and $Y = \{Y_t : t \geq 0\}$, the pathwise covariation of X and Y over $[0, t]$, denoted by $[X, Y]_t$, is defined as $[X, Y]_t = \lim_{n\to\infty} \sum_{k=0}^{n} (X_{ti/n} - X_{t(i-1)/n})(Y_{ti/n} - Y_{t(i-1)/n})$. In particular, if $X = Y$, we denote the quadratic variation of X just by $[X]_t$.
$\mathbb{E}[e^{sX}]$	If $X \sim \mathcal{N}(\mu, \sigma)$, the moment generating function of X at point $s \in \mathbb{R}$ is given by $\mathbb{E}[e^{sX}] = e^{s\mu + \frac{1}{2}\sigma^2 s^2}$.
a.s.	Almost surely.
i.i.d.	Independent and identically distributed.
SDE	Stochastic differential equation.
w.l.o.g.	Without lost of generality.
w.r.t	With respect to.

The resolution of the exercises, when presented, has a yellowish background.

Chapter 1

Introduction to Derivatives Pricing

1. Let $B(t,T)$ denote the cost at time t of a risk-free 1 euro bond, at time T. Assume that the interest rate is a deterministic function. Show that the absence of arbitrage requires that

$$B(0,1)B(1,2) = B(0,2).$$

First we prove that if $B(0,1)B(1,2) > B(0,2)$, then we would have an arbitrage. Then we construct our portfolio as follows:

- at $t = 0$, you agree to sell $B(1,2)$, which means that you receive $B(0,1)B(1,2)$ for that product. Buy $B(0,2)$. Then at time $t = 0$, you have $B(0,1)B(1,2) - B(0,2) > 0$.
- at $t = 2$, you receive your bond (the one that you have contracted at time $t = 0$), which means that you receive 1 €, and at the same time you need to give 1 €to the buyer of the bond $B(1,2)$

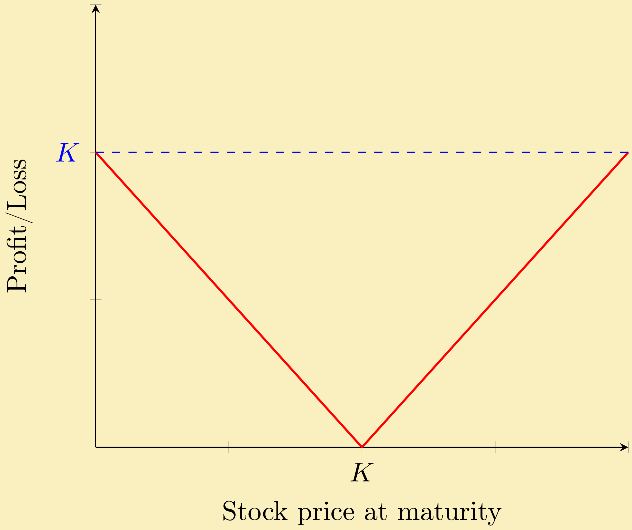
So with this strategy you would have a positive profit, with probability one, which would be an arbitrage possibility.
Now assume that $B(0,1)B(1,2) < B(0,2)$. Then we construct our portfolio as follows:

- You sell one bond with maturity 2, receiving $B(0,2)$, and you buy $B(1,2)$, for which you need to pay $B(0,1)B(1,2)$. So at time $t = 0$, you have $B(0,2) - B(0,1)B(1,2) > 0$.
- At time $t = 2$, you get 1 €(from the bond that you bought), and you need to pay 1 €(from the bond that you sold).

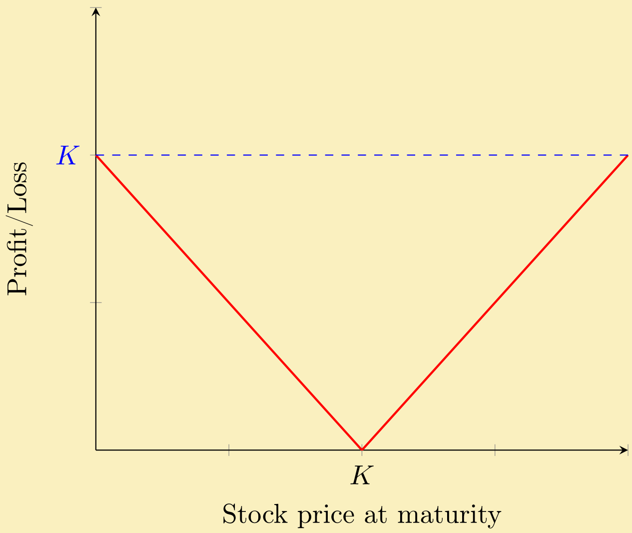
Again you end up with a positive profit, and consequently you have an arbitrage possibility.

2. For each of the following portfolios, draw the expiry payoff diagram and explain what view of the market holding this position expresses:

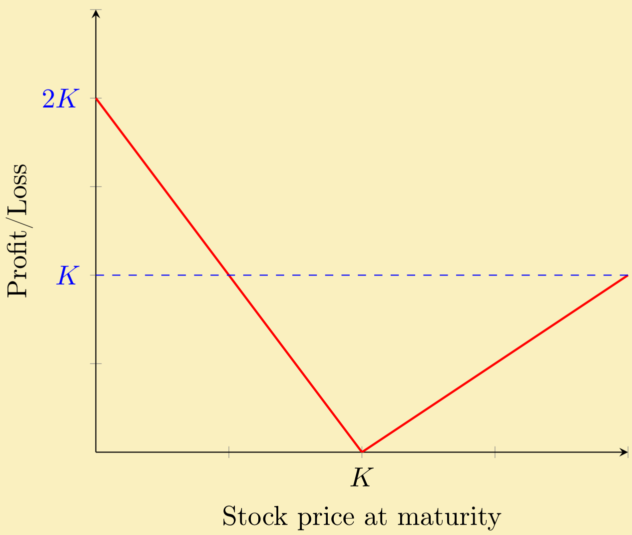
   a) One long call and one long put options, both with the same strike (*straddle* strategy).
   b) One short forward and two long calls, all with the same strike price.
   c) One long call and two long puts, all with the same strike price (*strip* strategy).
   d) One long put and two long calls, all with the same strike price (*strap* strategy).
   e) One long call with strike price K_2 and one long put, with strike price K_1.
   f) One long call with strike price K_1, one long call with strike price K_2 and two short calls, both with strike price $(K_1 + K_2)/2$ (*butterfly* strategy).


Profit/Loss
K
K
Stock price at maturity

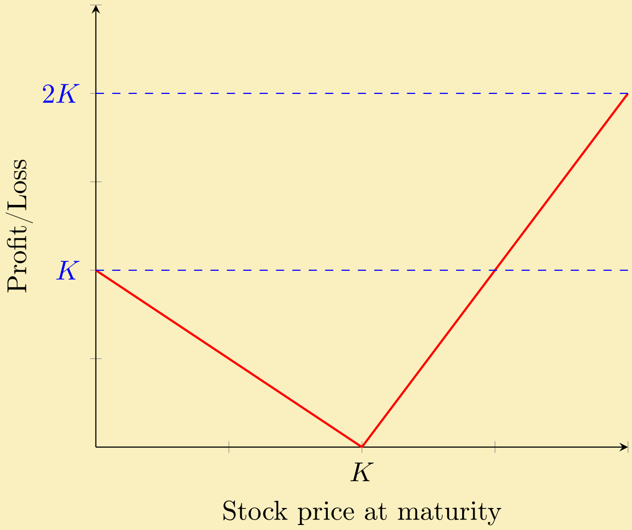
a)


Profit/Loss
K
K
Stock price at maturity

b)

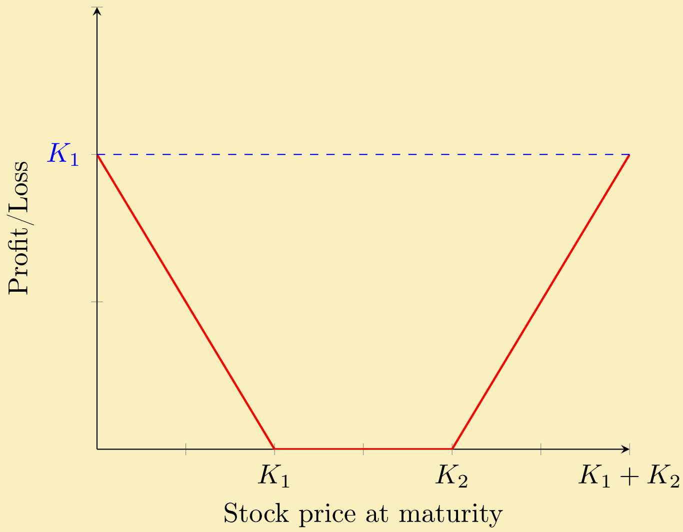

Profit/Loss
$2K$
K
K
Stock price at maturity

c)

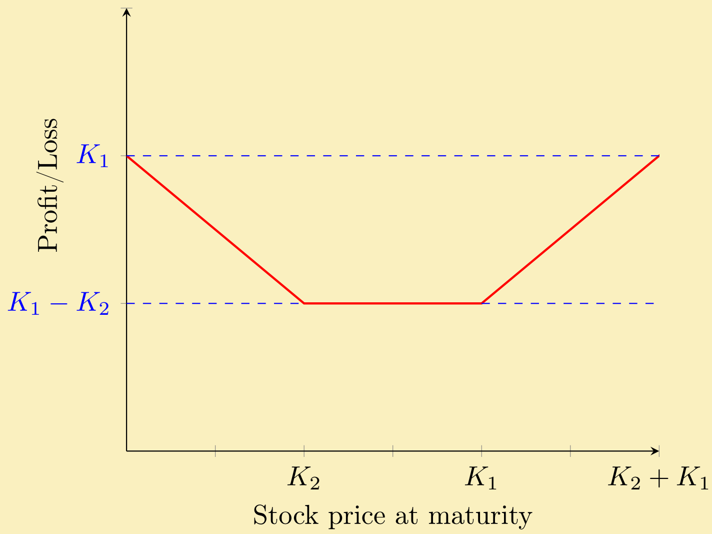


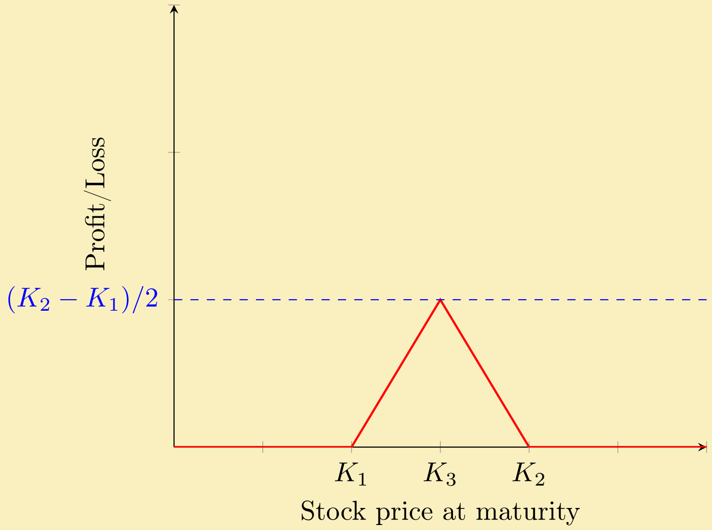
d)

e) – If $K_1 > K_2$:



– If $K_2 < K_1$





f)

3. Let S_0 denote the present price of a specified stock. In a forward agreement, one agrees at time 0 to pay the amount F at time T for one share of the stock that will be delivered at the time of the payment. Derive the non-arbitrage value of F.

If S_0 denotes the current value of the share, then in order to avoid arbitrage-possibilities one must have:

$$F = S_0 e^{rT}$$

as:

- if $F < S_0 e^{rT}$: then the forward price is too low; in that case you should have a long position in this contract. Then short sell one share; sell it by S_0 and put the money in the bank. At the maturity date buy the share, use part of the money that you have in the bank (which now worths $S_0 e^{rT}$) in order to pay the share and return the share to the short-seller. At the end you still earn money, without any initial investment, and therefore you have an arbitrage possibility.
- if $F > S_0 e^{rT}$: the forward price is too large and therefore you should have a short position in this contract. Then borrow S_0 to the bank and go to the market and buy the share by S_0. At the maturity of the contract sell this share by F and return $S_0 e^{rT}$ to the bank (your loan); as $F > S_0 e^{rT}$, you still earn free-risk money, without any initial investment, and therefore you have an arbitrage possibility.

4. Suppose that c_i is the price of an European call option with maturity T and strike price K_i, with $i = 1, 2, 3$. Assume that $K_1 < K_2 < K_3$, with $K_2 = \frac{K_1+K_3}{2}$. Consider a porfolio consisting of one long position with strike price K_1, one long position with strike price K_3 and two short positions with strike price K_2, each.

   a) Give the payoff at the maturity date T of this portfolio.

   b) Prove that $c_2 \leq 0.5(c_1 + c_3)$.

a) If you do not take into account the prices of these contracts, the payoff at the maturity of this porfolio is as follows:

- If $S_T < K_1$: options are not exercised. So your payoff is zero.
- If $K_1 < S_T < K_2$: You exercise option 1, and therefore your payoff is $S_T - K_1 > 0$; the maximum payoff occurs when $S_T = K_2$ and therefore is equal to $(K_2 - K_1)/2$.
- If $K_2 < S_T < K_3$: You exercise option 1 and the two options 2 are exercised by the holder. Then your payoff is $(S_T - K_1) - 2(S_T - K_2) = K_3 - S_T > 0$; the maximum payoff occurs when $S_T = K_2$ and therefore is equal to $K_3 - K_2 = (K_3 - K_1)/2$.
- If $S_T > K_3$: all the options are exercised. Then your payoff is: $(S_T - K_1) + (S_T - K_3) - 2(S_T - K_2) = 0$.

b) In all the situations the payoff is larger or equal to zero. Therefore in order to avoid an arbitrage possibility you have to assume that the gain of these contracts at time 0 (which is equal to $2c_2 - (c_1 + c_3)$) is negative, ie,

$$c_2 < 0.5(c_1 + c_3).$$

5. A two-year European call option at 90 Euros strike is quoted at 5 Euros and a two-year European call option at 95 Euros strike is quoted at 5.1 Euros. Design a trading strategy to lock profits (as the two prices allow arbitrage!).

It is clear that the second option is overpriced (a higher strike price should have a lower price). Therefore in order to have an arbitrage opportunity you should be the writer of the second option (short position) and the holder of the first one (long position). Then, as the maturities are the same:

i) If $S_2 < 90$: both options are not exercised; gain$=(5.1-5)e^{2r}$ (r is the annual interest rate).

ii) If $90 \leq S_2 \leq 95$: you exercise your option. Then you buy the share by 90 Euros and immediately you sell it by S_2, earning $0.1e^{2r} + (S_2 - 90)$.

iii) If $95 < S_2$: then both options are exercised. You buy one share by 90 and you immediately return it to the holder of the second contract, by 95. Then your profit is $0.1e^{2r} + 5$.

Then in all the situations you have a positive profit, and thus you have a clear arbitrage opportunity.

6. Consider the following combined strategy, defined on the same underlying asset: buy two European call options with strike price K_1 and K_3 (with $K_1 < K_3$) and sell two European call options with strike price $K_2 = 0.5(K_1 + K_2)$.

a) Plot the payoff of this combined strategy (usually known as *butterfly spread*).

b) Use put-call parity to show that the cost of a butterfly spread created from European puts is equal to the cost of a butterfly spread created from European calls.

b) Assume that the cost of these three calls is c_1, c_2 and c_3, whose strike prices are K_1, K_2 and K_3, respectively. Now assuming the put-call parity, it follows that if we denote the corresponding puts prices by p_1, p_2 and p_3, respectively, it follows that:

$$c_1 + K_1 e^{-rT} = p_1 + S_t$$
$$c_2 + K_2 e^{-rT} = p_2 + S_t$$
$$c_3 + K_3 e^{-rT} = p_3 + S_t$$

Therefore it follows that

$$(c_1 + c_3 - 2c_2) + (K_1 + K_3 - 2K_2)e^{-rT} = (p_1 + p_3 - 2p_2) + (S_T + S_T - 2S_T)$$

But as $K_2 = (K_1 + K_3)/2$, then it follows that

$$c_1 + c_3 - 2c_2 = p_1 + p_3 - 2p_2$$

ie, the cost of this strategy involving call options is the same if you use put options.

7. A European call option and an European put option on a stock that expires in one year have both a strike price of 44 Euros. The current stock price is 40 Euros and the one-year risk free interest rate is 10%. The price of the call is 10 Euros and the price of the put is 7 Euros. Design an arbitrage possibility.

Let us first check if the put-call parity holds, ie.,

$$c + Ke^{-rt} \stackrel{?}{=} p + S_0$$

In this case $c + Ke^{-rt} = 10 + 44e^{-0.10} = 49.8128$, whereas $p + S = 47$, meaning that the call is overpriced. Then you have the following arbitrage possibility: go short in the first contract and go long in the second contract. Then that means that at time 0 you receive $10 - 7 = 3$ Euros, which you put in the bank; at the end of the first year you own 3.3 euros. If $S_1 < 44$, then no one exercises his rights, and thus your net profit is 3.3; if $S_1 > 44$, both options are exercised, and therefore your net profit is also 3.3. In either cases you have an arbitrage opportunity.

8. Let $S(t)$ denote the price of a given security at time t. All of the following options have exercise time T and (unless stated otherwise) strike price K. Give the payoff at time t that is earned by an investor who:

a) owns one call having exercise price K_1 and has sold one put having exercise price K_2;

b) owns two calls and has sold one share of the security;

c) owns one share of the security and has sold one call.

9. A bond has face value 500 Euros, and coupons equal to 30 Euros, paid at the end of each year. The maturity date for such product is 5 years. Prove that the value of the bond is 528.142, assuming that the annually compounded interest rate is 5%.

10. A *chooser* option is an agreement where the owner of the option has the right to choose at a fixed *decision time* $T_1 < T_2$ (where T_2 denotes the maturity) whether the option is to be a call or a put, with a common strike price K, and remaining time to expiry $T_2 - T_1$.

    a) Let c and p denote, respectively, the value of the call and of the put underlying the option at time T_1. Argue that the value of the chooser option at time T_1 is given by $\max(c, p)$.

    b) Derive a relation between c and p.

    c) Prove that you can replicate this chooser option using a call with strike price K and maturity T_2, and a put option, with strike price $Ke^{-r(T_2-T_1)}$ and maturity T_1. Derive the payoff of this strategy at time T_2.

    d) Introducing the notation $C(t, T, C)$ (resp. $P(t, T, c)$) for the time t value of a call (resp. put) option with the expiry T and strike price c, then for any $t = 0, 1, \dots, T_0$, the time t value $V_{ch}(t)$ of the chooser option can be represented as

    $$V_{ch}(t) = P(t, T, c) + C(t, T_0, c(1+r)^{-(T-T_0)})$$

    or, equivalently,

    $$V_{ch}(t) = C(t, T, c) + P(t, T_0, c(1+r)^{-(T-T_0)}).$$

    e) If p_{T_0} denotes the price of the put option at the decision time T_0, prove that T_0 is given by:

    $$T_0 = p_{T_0} + \left[S(T_0) - \frac{c}{(1+r)^{T-T_0}} \right]^+$$

    f) How can you replicate a choose option using a call and a put option?

11. Consider a T multi-period binomial model, and let X denote the net profit of an European call option. Moreover, denote the fair price of such contract at time t by $\Pi(t, X)$, for $t \leq T$. Argue that

    $$\Pi(T, X) = X.$$

    Using arbitrage arguments, we easily prove the result. For example, if $\Pi(T, X) < X$, then we buy the call at time T; we get a stock, and we sell it right away, earning $X - \Pi(T, X)$, without any initial investment and with 100% sure of getting a positive profit. Similar argument stands on the other case.

12. Let X be the gain from a portfolio strategy with initial cost x, such $X \geq 0$ in every state of the world. Prove that $x \geq 0$.

13. Draw a diagram showing the profit of the following strategy: 1 long stock, 1 long call, with strike price K_1 and 2 short call, with strike price K_2.

14. A one-month European put option on a non-dividend-paying stock is currently selling for 2.5€. The stock price is 47€, the strike price is 50€, and the risk-free interest rate (continuously compounded) is 6% per annum. What opportunities are there for an arbitrageur?

    The arbitrage opportunity is as follows: borrow $47 + 2.5 = 49.5$ €to the bank, that you will pay in one-month (therefore you need to return back $49.5 \times 1 + \frac{0.06}{12} = 49.7475$. Then with this money buy the put and buy one stock. At the end of the month, one of two things may happen:

    – The stock price is lower than 50€: in that case you exercise the put-option, selling the stock by 50€. You return back to the bank 49.7475, having a positive net profit.

    – The stock price is larger than 50€: then you don't exercise your option but you go to the market and sell you stock, receiving more than 50€. Then you still have a positive net profit.

    In any case you have a positive gain, with zero investment, and thus you have an arbitrage opportunity.

15. A European call and put option on the same security both expire in three months, both have a strike price of 20 €, and both are traded for the price 3 €. If the nominal continuously compounded interest rate is 10% and the stock price is currently 25 €, identify an arbitrage.

We first check if the put-call parity holds in this case:

$$C + Ke^{-0.10\times\frac{3}{12}} = 3 + 20e^{-3} = 22.5062 \neq p + S(0) = 28$$

Thus the call-option is under-priced, so that in order to identify an arbitrage, we should buy a call and sell a put. (...)

16. A call and a put option over the same underlying asset whose present value is 110 Euros, with the same maturity and same strike price 100 Euros, are priced 12 and 5 Euros, respectively. Assuming there the interest rate is zero, design an arbitrage possibility. Would it still be possible to design an arbitrage possibility if the strike price of the call would be less than 100 Euros?

Using the put-call parity, we conclude that the call is undervalued and/or the price is overvalued. Therefore one may create the following arbitrage possibility: short-sell one stock, sell it in the market, and with the money buy one call and sell one put. So at $t = 0$ you have $110 - 12 + 5 = 113$. Now we examine what happens on expiration. If $S_T > K$, then you exercise your call, buying a stock by 100 euros, returning back to the short-seller. The net profit is then 3 Euros. If $S_T < K$, then the put is exercised. In that case you are obliged to buy the stock, paying 100 euros and returning it back to the short-seller. Anyway, your net profit is 3 Euros, no matter what happens.
If the strike price of the call would be less than 100 euros, then its price would be even cheaper, and for that reason the put would be even overvalued. So the arbitrage possibility would be still possible, and would be the same, with the same strategy (but with higher or equal net profit).

17. Consider two bank accounts, $B_1 = \{B_1(t), t > 0\}$ and $B_2 = \{B_2(t), t > 0\}$ such that:

$$dB_1(t) = rB_1(t)dt; \quad dB_2(t) = \mu B_2(t)dt$$

Furthermore let $S = \{S(t), t \geq 0\}$ denote the stock price of a certain asset.

a) Prove that if $\mu \neq r$ then one has arbitrage.

b) Consider now a call option and a put option on the same underlying asset, both with maturity T and strike price $K = S(0)e^{rT}$. Assume that the price of the call is c and the price of the put is p, with c and p given by:

$$c = p + x$$

for a certain amount x. Prove that if $x > 0$ one has arbitrage.

c) Show that the following is an upper bound for p:

$$p \leq S_0.$$

a) Assume, w.l.o.g., that $\mu > r$, and one-year continuously compounded. Then the following is an arbitrage possibility: borrow K euros of B_1, and invest in type B_2 assets, for one year, say (in fact time here is whatever you wish!). At the end of the one-year contract you have Ke^{μ} and you have to return Ke^r. So you have, with probability 1, a net profit of $Ke^r(e^{(\mu-r)} - 1) > 0$.

b) Clearly the put-call parity does not hold, but we can also check as follows: sell one call (receiving c). With that money buy one put, borrow $S(0) - x$ and buy one stock. At the end of the contract either $S(T) > K$ or not. If $S(T) > K$: the call is exercised but the put is not. Thus you sell your stock, receiving $K = S(0)e^{rT} = (S(0) - xe^{rT}) + xe^{rT}$, which is more than the money that you need to return to the bank. So overall you have a net profit of $xe^{rT} > 0$. On the other hand, if the stock price is bellow K: then the call is not exercised but you exercise your put. Then you sell the asset by K, and thus once again, following a similar reasoning to the previous case, you have a net profit of $xe^{rT} > 0$. Either cases you have an arbitrage possibility.

18. Suppose that a certain stock is currently worth 61€. Consider an investor that buy one call with a strike price equal to 55€, that costs 10€, buys another call with strike price equal to 65€, paying for such a call 5€ and sells two calls with a 60€ strike price, receiving 7€ for each of such call (assume that all the options have the same underlying asset and same maturity).

    a) Derive the profit from such a spread.
    b) Present a combination of puts, instead of calls, such that you have exactly the same payoff as you have with this combination of calls.
    c) Derive the price of such spread, using only puts.

19. The price of a European call which expires in 6 months and has a strike price of 30 is 2. The underlying stock price is 29, and a dividend of 0.50 is expected in two months and in five months. The annual interest rate is 10%. What is the price of a European put option that expires in 6 months and has a strike price of 30?

$T-t=6m=\frac{1}{2}; X=30; C=2; S_t=29; r=0.1; D_1=0.5; t_1-t=2m=\frac{1}{6}; t_2-t=5m=\frac{5}{12}$.
The put-call parity implies

$$C+Xe^{-r(T-t)}+D_1e^{-r(t_1-t)}+D_2e^{-r(t_2-t)}=P+S_t$$
$$2+30e^{-0.1\frac{1}{2}}+0.5e^{-0.1\frac{1}{6}}+0.5e^{-0.1\frac{5}{12}}=P+29$$

and, hence, $P=2.51$

20. Call options on a stock are available with strike prices of 15, 17.5, and 20 €, and expiration dates in 3 months. Their prices are 4, 2, and 0.5 €, respectively. An investor has two long positions in calls, one with strike price of 15 and the other 20 €, and has a short position in two call options with strike price of 17.5 €. Construct a table showing how profit varies with stock price.

Let S_t denote the stock price at time t. Then the stock price at maturity is denoted by S_T. The profit (without taking into account the prices of the options) of this strategy depend on S_T as follows:

- $S_T<15$: then neither of the options are exercised, so that the profit is equal to 0.0.
- $15<S_T<17.5$: the call whose strike price is 15 is the only to be exercised. Thus the profit is S_T-15.
- $17.5<S_T<20$: the calls with price prices 15 and 17.5 are exercised. Taken into account the short and long positions, the profit is: $S_T-15+2(17.5-S_T)=20-S_T$.
- $S_T>20$: all the calls are exercised. Then the profit is $S_T-15+2(17.5-S_T)+S_T-20=0.0$

Note that the price (at time $t=0$) of this strategy is $4+0.5-2\times 2=0.5$, which means that the investor loses money (loses 0.5 €) if the price of the stock at the maturity is smaller than 15.5 or larger than 19.5.

21. The present price of a stock is 50. The price of a European call option with strike price 47.5 and maturity 180 days is 4.375. The cost of a risk-free 1 euro bond 180 days is 0.98.

    a) Consider a European put option with price 1.025 (same strike price and maturity as the call option). Show that this is inconsistent with put-call parity.
    b) Describe how you can take advantage of this situation, creating an arbitrage possibility example.

a) Just plug-in the values in the put-call parity formula:

$$c+Ke^{-rT}=p+S(0)$$

From the price of the bond, and assuming continuously compounding, we can get the interest rate:

$$0.98\times e^{r\times 0.5}=1\Leftrightarrow r=0.040$$

(where r is the annual interest rate, as we considered 180 days to be equal to 0.5 (years)). Therefore:

$$4.375+47.5\times 0.98=50.925<1.025+50=51.025$$

which means that the call (put) is under(over) priced.

b) As the call is under-priced, your arbitrage strategy involves buying a call and sell a put. Therefore you proceed as follows:

   – Buy a call, sell a put, short-sell one stock and sell it in the market. So you are left with:

$$-4.375 + 1.025 + 50 = 46.65$$

   which you invest in a free-risk application, meaning that at the maturity that money worth 47.6019.

   – At maturity, one of two things may happen:
     – $S_t < 47.5$: then only the put is exercised, meaning that you are obliged to buy a stock, paying 47.5. You return the stock to the short-seller, and you get a net profit of $47.6019 - 47.5 = 0.1019$
     – Otherwise: the call is exercised. Then you buy one stock in the market, paying *only* 47.5. As in the other case, you return that stock to the short-seller, and you still get a net profit of 0.1019.

   So with probability one, you get a net profit of 0.1019, which leads to an arbitrage possibility.

22. Assume that a European call on a stock with strike price $K = 50$ and exercise date in six months sells for 4.75, and that a put, with same maturity, exercise date and on the same underlying stock, sells for 2.25. The price of the underlying asset today is 52, the continuously compounded risk-free interest rate equals 0.06 and assume that dividends in equal amounts of 0.73 are to be paid on 2 and 4 months.

   a) Design an arbitrage portfolio.

   b) Propose a put-call parity relation for the case of discrete dividends D_i, $i = 1, 2 \ldots, n$ at times $0 < t_1 < t_2 \ldots < t_n = T$, where T is the maturity of the contracts.

a) In this case we will prove that this is an arbitrage portfolio because the call is overpriced when compared to the put. Thus we propose the following arbitrage argument:

   - sell such a call, buy the put and buy one of the underlying stocks. In order to be able to perform such operations, we need to borrow $52 - 4.75 + 2.22 = 49, 47$, that we invest in the bank.
   - At the end we need to return back $49.47e^{0.06*6/12} = 50.977$ As we have a stock, we receive the dividends, meaning that in the maturity we have $0.73 \times (e^{0.06*4/12} + e^{0.06*2/12}) = 1.482$.
   - At the maturity:
     – either $S(T) < K$ (and in that case you exercise the put; you sell your stock, receiving 50. As we have 1.482 due to the dividends and your need to return 50.977, you have a net profit of 0.505)
     – or $S(T) \geq K$ (and in that case you have exactly the same net profit, as the owner of the call exercises his option, and you are obliged to sell him the stock by 50).

   So with probability one, you have an arbitrage portfolio.

b) The difference to the standard put-call parity is that we have the dividends, which are received by the person who owns the stock (and thus is also the buyer of a put-option). Therefore the put-call parity at time 0 in this case is:

$$c(0) + Ke^{-rT} = p(0) + S(0) - \sum_{i=1}^{n} D_i e^{-rt_i}$$

   where $c(0)$ ($p(0)$) is the price of the call (put) at time 0, and assuming a continuously compounded interest rate. Also the interest rate, r, needs to be per same unit of time as T and t_i (days, months, or so).

23. Consider an European put and a call, with the same underlying asset whose current price is $S(0)$, same maturity T, and same strike price K. Let p and c denote, respectively, the price of the put and the call. Furthermore, assume that it is known that a dividend D will be received at time $t < T$. In this case, the put-call parity can be written as follows:

$$c + Ke^{-rT} + De^{-rt} = p + S(0)$$

   Identify the arbitrage opportunity for the following values: $K = 20$, T is equal to 3 months, $c = p = 3$, $r = 10\%$ (per year), $S(0) = 19$ and one dividend of 1 Euro will be received in one month.

In view of the put-call parity, the call is overpriced. The arbitrage argument goes as follows:

- Sell one call, buy one put, borrow 19 euros from the bank
- At time 1, we receive 1 euro
- At maturity, the following happens:
  - If $S(T) < K$: the put is exercised. Then the payoff is $K + e^{\frac{0.1}{12}\times 2} - 19e^{\frac{0.1}{12}\times 3} = 1.54$
  - If $S(T) \geq K$: the call is exercised. Then the payoff is $K + e^{\frac{0.1}{12}\times 2} - 19e^{\frac{0.1}{12}\times 3} = 1.54$

  which ends the argument (proving that with probability 1, we have a positive profit)

24. A digital (K, T) call option gives its holder 1 unit of money at expiration time T if $S_T > K$, and 0 otherwise. A digital (K, T) put option gives its holder 1 unit of money at expiration time T if $S_T < K$, or 0 otherwise. Let $C_1(t)$ and $C_2(t)$ be the costs of such call and put options on the same underlying asset and same maturity, at time t. Derive a put-call parity relationship between $C_1(t)$ and $C_2(t)$.

If you buy one put and one call of such type, with probability 1, the payoff of such combination 1, which, discounted back in time (from T to t), is equal at time t to $e^{-r(T-t)}$, where r is the constant rate (here assumed continuously compounded). Therefore the price of such portfolio has to be equal to this payoff, meaning that:

$$C_1(t) + C_2(t) = e^{-r(T-t)}$$

25. Suppose that c_1, c_2 and c_3 are the prices of European call options with strike prices K_1, K_2 and K_3 (with $K_1 < K_2 < K_3$), respectively, where $K_3 - K_2 = K_2 - K_1$, all with the same maturity T and same underlying asset. Derive an inequality involving the prices of these contracts c_1, c_2 and c_3.

Assume that you buy the European options with strike price K_1 and K_3, and you sell two options with strike price K_2. Then you initial investment is $-2c_2 + (c_1 + c_3)$. At maturity, you have the following payoff:

$$\max(S_T - K_1, 0) + \max(S_T - K_3, 0) - 2\max(S_T - K_2, 0)$$
$$= \begin{cases} 0 & S_T < K_1 \\ S_T - K_1 & K_1 \leq S_T < K_2 \\ S_T - K_1 - 2(S_T - K_2) = 2K_2 - K_1 - S_T & K_2 \leq S_T < K_3 \\ S_T - K_1 + S_T - K_3 - 2(S_T - K_2) = 2K_2 - K_1 - K_3 & S_T \geq K_3 \end{cases}$$

If $K_3 - K_2 = K_2 - K_1$ then it follows that $2K_2 - K_1 - K_3 = (K_2 - K_3) + (K_2 - K_1) = 0$. Therefore as $\max(S_T - K_1, 0) + \max(S_T - K_3, 0) - 2\max(S_T - K_2, 0) \geq 0$ with probability one, it follows that the price of this portfolio has to be positive, meaning that $-2c_2 + (c_1 + c_3) \geq 0$.

26. Prove that with no dividends prior to expiration, the value of an American put option, P_0, is restricted by:

$$P_0 \leq C_0 - S(0) + K$$

where C_0 is the value of the corresponding American call option, K is the strike price and $S(0)$ is the price of the underlying asset.

Consider portfolio A, which contains a call and K invested in the riskless asset, and portfolio B, which contains a put option together with the stock. The payoff of these portfolios is the following:

Portfolio	Value at date t (if the put is exercised)	Value at maturity $S(T) < K$	$S(T) \geq K$
A: $C_0 + K$	value of the call at $t + Ke^{rt}$	Ke^{rt}	$(S(T) - K) + Ke^{rt}$
B: $P_0 + S(0)$	K	K	$S(T)$
	$V_A(t) > V_B(t)$	$V_A(t) > V_B(t)$	$V_A(t) > V_B(t)$

Thus the price of portfolio B needs to be larger than the price of portfolio A, by the no-dominance principle.

27. Consider a European call option and a put option on a stock each with a strike price of 22€, with maturity 6 months. The price of the call is 4€ and the price of the put is 3€. The 6-month risk free rate of interest is $r = 10\%$ (which you may assume continuously compounded). The current stock price is $S_0 = 20$€. Identify the arbitrage strategy.

Chapter 2

Binomial Model

1. Suppose that in each year the cost of a security either goes up by a factor of 2 or down by a factor of 1/2 (i.e., $u = 2$ and $d = 0.5$). Assume a continuously compounding euro interest rate of 4%.

   a) If the initial price of the security is 100, determine the non-arbitrage cost of an European call option to purchase the security at the end of two periods for a strike price of 150.

   b) Assume that your option is American. Should you exercise early?

a) In this case, given that $u = 2$, $d = 0.5$ and $r = 4\% = 0.04$, then $q = \frac{1+r-d}{u-d} = 0.36$. In addition:

S_0	S_1	S_2	gain
100	200	400	$\Phi(S_2) = 250$
	50	100	$\Phi(S_2) = 0$
		25	$\Phi(S_2) = 0$

Then the price of the option at the end of the first year if the security price goes up is equal to:

$$\frac{1}{1.04}(0.36 \times 250) = 86.54$$

and if it goes down then it is zero. Therefore the cost of this call is:

$$\frac{1}{1.04}(0.36 \times 86.54) = 29.96.$$

b) If this option is American, instead of European, then if at the end of the first year it goes up, then the option should be exercised, as in that case the payoff is $50 (= 200 - 150)$, whereas the expected gain if the option is postponed is $86.54 > 50$. Therefore it should be postponed.

2. A stock price is currently 20 Euros, but it is known that in two months it can be either 25 Euros or 18 Euros. Let S denote the value of the stock price in two months. Suppose that there is a derivative that pays off S^2 and that the interest rate is 12% per annum.

   a) Determine the free-arbitrage pricing of this derivative.

   b) Admit that you have 43 of these shares and that you sign this contract so that at maturity you have to pay S_T^2. Prove that in this case you are risk-free.

a) If we denote by $\Pi(0; X)$ the derivative pricing, then one can prove that $\Pi(0; X) = \frac{0.343 \times 25^2 + 0.657 \times 18^2}{1.02} = 418.9$.

b) For such a portfolio, the value of it is unchanged (i.e., if the stock price goes up then we get $43 \times 25 - 25^2$, which is equal to the value of the portfolio if it goes down, $43 \times 18 - 18^2$). Thus it is risk-free, under this model.

3. Consider a two-period binomial model, with one risky asset. Now assume that you have two kinds of derivatives: the first one is a plain European call option; for the second one the payoff is a function of an average of the prices, i.e., if $S_{av} = \frac{1}{2}(S_0 + S_2)$, then the payoff is

$$(S_{av} - K)^+$$

   a) Without any calculations, explain which derivative should have a higher price.
   b) Assuming that $u > 1$ and $ud > 1$, is it possible to establish which derivative has higher arbitrage-free price?
   c) Derive the arbitrage-free price for both derivatives, assuming the following numerical values: $u = 4, d = 0.5, r = 1,\ K = S_0 = 4$.

4. Suppose that $T = 2$, and that the initial price of the stock is $S_0 = 100$. Assume that it the price may go up by a factor $u = 1.5$ or by a factor $d = 1.1$, and that the strike price, K, is equal to 150. Assume a risk-free rate of 20%. Prove that the non-arbitrage price of this call is 7.16.

5. Consider a 2-year European put, with a strike price equal to 52 Euros on a stock whose current price is 50 Euros. We assume that there are two time steps of one year, and in each time step the stock price either goes up by 20% or moves down by 20%. We assume also that the risk-free interest rate in 5% per annum.

   a) Find the arbitrage free value of the option.
   b) Determine the replicating portfolio.
   c) Find the arbitrage free value of such an American put option and describe the optimal policy.
   d) Now suppose that the interest rate is not fixed: in the first period it is 5%, while in the second period depends on the state of the nature. If the stock goes up, the interest rate is 5%, whereas if it goes down, the interest rate is 10%. Is the market arbitrage-free?

6. Consider the 2-period binomial model, with parameters $u = 2$, $d = 1/2$ and $r = 1/4$. Compute the standard deviation of the logarithmic return of the stock, i.e., compute the volatility of $X = \log S_1/S_0$.

7. In this problem we work with the standard 3-period binomial model, with the same data as in the previous exercise; moreover, assume that $S_0 = 4$. Now consider an American option not on the values of the underlying stock $\{S_n\}$ but on the following product:

$$G_n = \left(\frac{1}{n+1} \sum_{i=0}^{n} S_i - S_n \right)$$

   What is the optimal exercise strategy?

8. Consider a two-year European put-option, with a strike price of 52€ on a stock whose current price is 50€. We suppose that there are two time steps of one year, and in each time step the stock price either moves up, by a proportional amount of 20%, or moves down, by a proportional amount of 20%. We also suppose that the risk-free interest rate is 5% per annum (you may assume 1-year compounded).

   a) Derive the fair price of this option.
   b) Now assume that this option is an American one. Describe then what is the optimal decision. In particular, when is early-exercise optimal? Derive also the value of this put.

   a) In the case of the European put-option, the price of it is 4.2517.

   b) If it is an American option, then you should exercise early if the price of the stock goes down in the first year, and its price is 5.1360.

9. Consider a option such that the payoff at the maturity is given by $\max(0.5(S_0 + S_T) - K, 0)$, where S_0 (S_T) denotes the stock price at initial time (maturity) and K is the strike price. Assume a binomial model, with $S_0 = 40$, $u = 1.3$, $d = 0.8$, $K = 45$ and 2 years maturity. The interest rate changes with time, in the following way: in the first year it is 3% and in the second year it is 4%.

   a) Derive the price of this option.
   b) Assume that the option can be exercised at the end of the first year or at maturity. When will it be optimal to exercise?

a) This is a typical pricing exercise except that the interest rate is not the same during the all period. Thus one need to compute different martingale measures and update the price using different interest rates. In fact, we have the following:

$$S_0 = 40, S_u = 52, S_d = 32, S_{uu} = 67.6, S_{ud} = S_{du} = 41.6, S_{dd} = 25.6$$

and the payoff in the maturity is $0.5 \times (40 + 67.6) - 45 = 8.8$ if the stock prices goes up twice; otherwise it is zero.

As the interest rate changes in the second year, also the martingale measure changes. So we denote by α_i the martingale probability relative of the ith period, with $i = 1, 2$. Then:

$$\alpha_1 = \frac{1 + r - d}{u - d} = \frac{1.03 - 0.8}{1.3 - 0.8} = 0.46; \quad \alpha_2 = \frac{1.04 - 0.8}{1.3 - 0.8} = 0.48$$

Thus the price of this option at $t = 0$ is just:

$$\Pi(0) = \frac{1}{1.03} \frac{1}{1.04}(0.46 \times 0.48 \times 8.8) = 1.81389.$$

b) If the price goes up in the first year, and if we exercise then the payoff would be $0.5 \times (40 + 52) - 45 = 1$. If we postpone the exercise, then we get, in average and already discount back, $\frac{1}{1.04} 0.48 \times 8.8 = 4.06154 > 1$. Thus it is optimal to exercise only at maturity, and only if there are two consecutive ups.

10. Consider a two-year European put, with a strike price of 52 € on a stock whose current price is 50 €. We suppose that there are two time steps of one year, and in each time step the stock price either moves up by a proportional amount of 20% or moves down by a proportional amount of 20%. Assume that the risk-free interest rate is 5%. Derive the value of the put. Would you exercise this option before the maturity, if allowed?

11. You are interested in stocks of a company HAL currently priced at 90 euros. The stock price is expected to either go up by 25% or down by 20% each six months. The annual risk free interest rate is 5%. Your broker now calls you with the following offer: After 6 months you can choose whether or not to buy a call option on HAL's stocks with 6 months maturity (i.e. expiry is 12 months from now). This option has a strike price equal to 85 euros, and costs 10 euros.

    a) Derive the fair price of this *compound* option.

    b) Now imagine that you do not have any choice after 6 months, you have to buy the option, paying the fixed price of 10 euros. What is then the value of the contract?

12. A stock is currently sold at 90 Euros, and it is expected to go up by 15% or down by 20%. Consider a put-option, with maturity 1.5 years, and strike price 90. Using a 3-step binomial model, for which the interest rate is 0.75% (per semester), derive the optimal strategy (i.e., derive when the owner of the put should exercise it, optimally).

    - Martingale probabilities: $Q = (0.59; 0.41)$;
    - The put option should be exercised before maturity if the stock price goes down in the first two periods (when the stock price is 57.6).
    - If the stock price goes up in the first two periods, hitting the value 119.025, it is indifferent between stopping or continuing. In both cases the payoff is zero.
    - The price of this option is (approximately) 9.61.

13. Consider a two-period binomial model for the stock price with both periods of length one year. Let the initial stock price be S(0) = 100 and assume that the stock pays no dividends. Let the up and down factors be u = 1.25 and d = 0.75, respectively. Let the continuously compounded interest rate be r = 0.05 per annum. Roger is interested in purchasing a chooser option with the provision that he can choose if the option is a put or a call after one year. The strike for this option is 100 and the expiry date is two years. Using the above binomial tree, find the price of the chooser option.

14. Suppose that the stock price is currently 20, and that in the next period its price will either increase/decrease by 50%. Consider a put option, with an exercise price of 18, with maturity 12 months, currently sold at 0.86. Assuming a 2 period-model, derive the corresponding interest rate.

If r denotes the interest rate for a 6-months period, discretely compounded. Then it follows that the martingale probability Q is given by $Q = (0.5 + r, 0 - 5 - r)$ and that the expected payoff of such option is:

$$\Pi(0) = \frac{1}{(1+r)^2} E^Q[(K - S(2))^+] = \frac{1}{(1+r)^2} \left(3 \times 2 \times (0.5 + r)(0.5 - r) + 13 \times (0.5 - r)^2\right)$$
$$= \frac{1}{(1+r)^2} \left(4.75 - 13r + 7r^2\right)$$

Equating this to the actual price of the option, gives:

$$\frac{1}{(1+r)^2} \left(4.75 - 13r + 7r^2\right) = 0.86 \Leftrightarrow 6,14r^2 - 14,72r + 3,89 = 0$$

whose roots are 0,30 and 2,09. In view of the condition for non-arbitrage, the only admissible solution is $r = 0,30$, i.e., the 6-months compound interest rate is 30%.

15. Consider a binomial model for the dynamics of an asset price, that is currently worth 100 Euros (S_0), but that may go up, by a factor of 25% or may go down, by a factor of 20%. Assume that we write an European call option, with strike price 100, with maturity $T = 3$, and that the interest rate per period is equal to 0.01.

    a) Derive the non-arbitrage price of such contract.

    b) Assume the contract as before, but now use maturity $T = 2$. Also assume that the interest rate is not constant; it changes along time, in the following way: in the first period is equal to 0.01, increasing then to 0.015. What is the price of the contract in such situation? And what would happen if in the last period the interest rate is 0.030?

a) Let X denote such derivative, and Φ the contract function. The associated martingale probability is $Q = (\frac{1.01-0.8}{1.25-0.8} = 0.4667, 1 - 0.4667)$ and thus

$$\Pi(0; X) = \left(\frac{1}{1.01}\right)^3 E^Q[\Phi(S_3)]$$
$$= \frac{95.31225 \times 0.4667^3 + 25 \times 3 \times 0.4667^2 \times (1 - 0.4667)}{1.01^3} = 17.859.$$

b) In the first case, we can still price but now we have to use backwards definition. But beware that also the martingales probability change along time. So for the second period, we use the following:

$$Q = (\frac{1.015 - 0.8}{1.25 - 0.8} = 0.478, 1 - 0.478)$$

Therefore if at the first step the price goes up, the price of such option at that node is:

$$\frac{1}{1.015} \times 56.25 \times 0.478 = 26.49$$

The other node the price is zero. Then at time 0 we have the following price:

$$\Pi(0; X) = \frac{1}{1.01} \times 26.49 \times 0.4667 = 12.241$$

In case the interest rate increases in the second period, then the price of such contract also increases.

16. For a stock price $S = \{S_t\}$, assume a binomial model with $S_0 = 40$, $u = 1.3$ and $d = 0.8$. There is a put-option defined on this stock, with strike price $K = 45$, with maturity equal to $T = 3$ years. Assume that the annual interest rate is 5%.

    a) Determine the price of this put at the time $t = 0$.

    b) Derive the price of this option if you assume that the holder of the option can exercise at any moment until the maturity.

    c) What would be the price of an equivalent call-option (i.e., a call option defined on the same underlying asset, the same maturity and the same strike price as the put)? Assume, instead, that the call-price is 10€. Design an arbitrage possibility.

a) Using the martingale pricing formula, the price of this put at time 0 is equal to:

$$\pi^p(0) = \frac{1}{1.05^3} \sum_{i=0}^{3} \binom{3}{i} \alpha^i (1-\alpha)^{3-i} F(u^i d^{3-i} S_0)$$

where $F(u^i d^{3-i} S_0) = \max(45 - u^i d^{3-i} S_0, 0)$ and

$$\alpha = \frac{1.05 - 0.8}{1.3 - 0.8} = 0.5$$

Thus $F(d^3 S_0) = 24.52$, $F(d^2 u S_0) = 11.72$ and zero, for other cases. Thus:

$$\pi^p(0) = \frac{1}{1.05^3}(\binom{3}{2} \times 0.5^3 \times 11.72 + \binom{3}{3} \times 0.5^3 \times 24.52) = 6.44.$$

b) We need to compute, for each node, the payoff if we stop and the expected payoff if we continue, proceeding backwards. If we go through the tree, the only cases where the payoff in case we stop and exercise is larger than the expected payoff of continuing is either the stock market price decreases in the first period or it decreases twice in a row, meaning in fact that the best strategy is to exercise before maturity if at the end of the first period the price went down. Otherwise the option should only be exercised in the maturity. For this strategy, the price of this contract is 7.46, higher than the corresponding European option.

c) Let c (p) denote the price of a call (put) option. Then according to the put-call parity

$$c = p + S_0 - Ke^{-rT} \Leftrightarrow c = 10 + 6.44 + 40 - 45e^{-0.05\times 3} = 7.71$$

If, instead, the call-price is 10 €, then the call is overpriced, meaning that one wants to sell it. In that case we have an arbitrage possibility, as follows: sell the call by 10 and buy a put, paying 6.44. Borrow $40 - 10 + 6.44 = 36.44$ from the bank, and use 40 in order to buy a stock. At the maturity, you need to return to the bank $36.44e^{0.05\times 3} = 42.34$ Moreover, one of two things happen:

- $S_T < 45$: the call is not exercised but the put is. Then you sell your stock, receiving 45. You return 42.34 €to the bank, and you have a net profit of $45 - 42.34 > 0$
- $S_T \geq 45$: in that case only the call is exercised. Then you are obliged to sell your stock, paying 45 €, and you have a net profit of $45 - 42.34 > 0$

Then with probability 1 you get an arbitrage situation.

17. Consider a binomial model with 3 periods, with $u = 1.25, d = 0.8$ and $r = 1.07$ (per period). Assuming an European call option whose current underlying asset price is $S_0 = 400$ and strike price equal to $K = 375$, derive its price.

Martingale probabilities: $Q = (0.6; 0.4)$
Price$=\frac{1}{1.07^3}\left[0.6^3 \times 406.25 + 3 \times 0.6^2 \times 0.4 \times 125\right] = 115.71$

18. Consider two dates, $t_1 < t_2$, and the following contract: at time t_1 the holder receives, at no extra cost, a call option with expiry date t_2 and strike price S_{t_1}. Using a two-period binomial model, prove that the fair price of this option at time zero is:

$$\Pi(0; X) = \left(\frac{1}{r}\right)^{t_2 - t_1} S_0 u^2 p_2, \text{ with } p_2 = \frac{r-d}{u-d},$$

assuming the necessary conditions for r, u and d.

19. Consider the following option: the payoff at maturity T is $X 1_{S_T > K}$, where X is pre-speficied. Assume a binomial model, such that:

$$d < e^{r\Delta t} < u$$

where Δt denotes the length of each time step. Find the value of this option at time 0.

20. Consider an American put option, with $S(0) = 100, K = 100, u = 1.25, d = 0.80, r = 0.07$ (per period), with 2 periods. Derive the price of this option and the optimal strategy.

21. Consider a two-period binomial model, where each period is one year. The current price of the non dividend-paying stock is 20€, with $u = 1.2840$, $d = 0.8607$. Assume that the 1-year risk-free interest rate is 5%. Calculate the price of an American call option on the stock with a strike price of 22€.

Chapter 3

Probability Theory

1. A (discrete-time) stochastic process $\{Y(t), t \in \mathbb{N}\}$ on $(\Omega, \{\mathcal{F}_t\}_{t\in\mathbb{N}}, P)$ is a Markov process if

$$P(Y_{n+1} \in A|\sigma(Y_j, j \leq n)) = P(Y_{n+1} \in A|Y_n), \quad \forall A \in \Omega.$$

Do you think that any Markov process is a martingale? If you believe so, give some informal explanation; if not, provide a counterexample.

Consider the following process: $\{S_n, n \in \mathbb{N}_0\}$, where $S_n = \sum_{j=1}^{n} X_j$, where $\{X_n, n \in \mathbb{N}\}$ is a sequence of i.i.d. Bernoulli variables. Then it follows that $S_{n+1} = S_n + X_{n+1}$, and therefore this is trivially a Markov process, but $\mathbb{E}[S_n|\mathcal{F}_m] \neq S_m$ (where $\mathcal{F}_m$ denotes the sigma-algebra generated by the process until time m), unless $\mathbb{E}[X] = 0$ (i.e, unless $P(X = 0) = 1$).

2. Assume that on the probability space $(\Omega, \mathcal{F}, P)$ there is a filtration $\{\mathcal{F}_t, t \in \mathbb{N}\}$, which is an indexed family of σ-algebras on Ω, such that $\mathcal{F}_t \subseteq \mathcal{F}$, $\forall t$. Now define the process $Y = \{Y_t, t \in \mathbb{N}\}$ by:

$$Y_t = \mathbb{E}[X|\mathcal{F}_t]$$

Prove that $\{Y_t, t \geq 0\}$ is a $(\mathcal{F}, P)$-martingale. What can you say about any martingale M defined on a compact interval $[0, T]$, with $T < \infty$?

The result follows trivially from the Tower property plus the fact that Y_t is $\mathcal{F}_t$-adapted, as it is a deterministic function of a process that is trivially adapted to its natural filtration $\mathcal{F}$. Moreover, Y is also P-integrable if the expected value of X is also finite (which we assume to hold).
Finally, using this result, it follows that for any martingale M defined on a compact set $[0, T]$ (with T finite), it follows that:

$$M_t = \mathbb{E}[M_T|\mathcal{F}_t]$$

for any $t \leq T$.

3. Let X and Y be two independent, symmetric random variables in $\mathcal{L}^2$. Prove that in this situation the following holds:

$$\mathbb{E}[(X + Y)^2|X^2 + Y^2] = X^2 + Y^2.$$

In view of the symmetry of Y, it follows that $Y \overset{d}{=} -Y$, and thus $X + Y \overset{d}{=} X - Y$ and $(X + Y)^2 \overset{d}{=} (X - Y)^2$. Therefore

$$\begin{aligned}
\mathbb{E}[(X + Y)^2|X^2 + Y^2] &= \mathbb{E}[(X - Y)^2|X^2 + Y^2] \quad \Leftrightarrow \\
\mathbb{E}[X^2 + Y^2 + 2XY|X^2 + Y^2] &= \mathbb{E}[X^2 + Y^2 - 2XY|X^2 + Y^2] \quad \Leftrightarrow \\
\mathbb{E}[XY|X^2 + Y^2] &= 0
\end{aligned}$$

Thus it follows that $\mathbb{E}[(X + Y)^2|X^2 + Y^2] = \mathbb{E}[X^2 + Y^2 + 2XY|X^2 + Y^2] = X^2 + Y^2$.

4. Let $\{\epsilon_j, j \in \mathbb{N}_0\}$ be a sequence of i.i.d. Bernoulli random variables, with $P(\epsilon_i = 1) = 0.5$. Define another sequence of random variables as follows:

$$\begin{aligned}
Z_0 &= 1; \\
Z_n &= 2\epsilon_n Z_{n-1}, \quad n \geq 1
\end{aligned}$$

a) Prove that the sequence $\{Z_n, n \geq 0\}$ is a martingale with respect to the usual filtration.

b) Define $\tau = \min\{n : Z_n = 0\}$. Prove that $P(\tau < \infty) = 1$ and that τ is a stopping time to the sequence Z.

c) Derive $\mathbb{E}[Z_\tau]$ and comment the result, in the light of martingale properties.

a) We note first that $Z_n = 2^n$ with probability 0.5 or $Z_n = 0$, and thus $\mathbb{E}[Z_n] < \infty$. So $Z = \{Z_n, n \in \mathbb{N}\}$ is an integrable process. Regarding the martingale property:

$$\mathbb{E}[Z_n|Z_{n-1}] = \mathbb{E}[2Z_{n-1}\epsilon_n|Z_{n-1}] = 2Z_{n-1}\mathbb{E}[\epsilon_n] = 0$$

b) We have that $\tau < \infty$ with probability 1 because $\tau \sim Geo(0.5)$. τ is also a stopping time to Z as

$$\{\tau = n\} \Leftrightarrow \{Z_i = 2^i \wedge Z_n = 0\} \in \mathcal{F}_n,$$

where $\mathcal{F}$ is the natural σ-algebra of Z.

c)

$$\mathbb{E}[Z_\tau] = \mathbb{E}[0] = 0$$

but on the other side

$$\mathbb{E}[Z_\tau|Z_{\tau-1}] = 0 \neq Z_{\tau-1}]$$

meaning that the martingale property does not hold for stopping times. Recall the optional sampling theorem...

5. Show that the class $\mathcal{F}$ of subsets A of R such that A or A^c is discrete (finite or countably infinite) is a σ-algebra. Define on $\mathcal{F}$ a set function P by $P(A) = 0$ if A is discrete and $P(A) = 1$ if A^c is discrete. Show that P so defined is a probability measure.

In order to be a σ-algebra the following conditions are required:

- The empty set $\in \mathcal{F}$, which holds because the empty set is, by definition, countable.
- If $A \in \mathcal{F}$, then it means that it is countable (and in that case A^c is not countable, and $(A^c)^c = A$ is countable; thus in this case $A^c \in \mathcal{F}$), or then it means that A^c is countable (and thus it belongs to $\mathcal{F}$).
- If $A_i \in \mathcal{F}$, then $\cup_i A_i \in \mathcal{F}$. To see why this holds, remark that if A_i is countable, for all i, then its countable union is also countable and therefore $\cup_i A_i \in \mathcal{F}$. Now if at least one of the A_i is not countable, then the union is not countable, and therefore its complement is countable, and thus it belongs to $\mathcal{F}$.

So $\mathcal{F}$ so defined is indeed a σ-algebra.
Finally one needs to prove that the function P is a probability, i.e., one needs to prove:

- $P(\Omega) = 1$ as Ω is the complement of a discrete set (the empty set)
- If $A_i \cap A_j$ =empty set, for $i \neq j$, then if they are all countable, it follows that $P(A_i) = 0$; moreover, $\cup_i A_i$ is also countable and thus

$$P(\cup_i A_i) = 0 = \sum_i P(A_i) = \sum_i 0 = 0$$

Now assume that there K sets that are not countable. Then we know that its union is not countable, and therefore $P(\cup_{i\in K} A_i) = 1$. On the other side,

$$\cup_i A_i = \cup_{i\in K} A_i \cup_{i\notin K} A_i$$

and $\cup_{i\in K} A_i$ is also countable. Thus:

$$P(\cup_i A_i) = P(\cup_{i\in K} A_i) + P(\cup_{i\notin K} A_i) = 0 + 1 = 1.$$

6. Consider a sequence $X = \{X_n, n \in \mathbb{N}\}$, and let $\sigma(X_1, \ldots, X_n) = \mathcal{F}_n$, such that X is a submartingale, i.e., $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$. Moreover, let $A_0 = 0$ and:

$$A_n = \sum_{k=1}^{n} (\mathbb{E}[X_k|\mathcal{F}_{k-1}] - X_{k-1}) \quad M_n = \sum_{k=1}^{n} (X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}])$$

a) Prove that $\{M_n, n \in \mathbb{N}\}$ is a martingale.

b) Prove that $A_{n+1} \geq A_n$, for all n. The r.v. A_n is $\mathcal{F}_{n-1}$-measurable? And $\mathcal{F}_n$-measurable?

a) First of all we note that for each n, M_n is $\mathcal{F}_n$ measurable, because it is a sum of n r.v. that, by construction, are $\mathcal{F}_k$ measurable, for $k \leq n$, as X_k and $\mathbb{E}[X_k|\mathcal{F}_{k-1}]$ are $\mathcal{F}_k$ measurable. Moreover,

$$\mathbb{E}[|M_n|] \leq \sum_{k=1}^{n} \mathbb{E}|X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}]| \leq n \max_{k=1,\ldots,n} \mathbb{E}|X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}]|$$
$$= n \max_{k=1,\ldots,n} \mathbb{E}(X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}]) < \infty$$

as $\{X_k\}$ is a submartingale (thus $X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}] > 0$ with probability 1, and so $\mathbb{E}[X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}]] > 0$). Moreover, from the fact that $\{X_k\}$ is a submartingale it also follows that $\mathbb{E}[X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}]] < \infty$.
So we just have to check the equality in terms of expected values:

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \mathbb{E}[M_n|\mathcal{F}_n] + \mathbb{E}[X_{n+1}|\mathcal{F}_n] - \mathbb{E}[E(X_{n+1}|\mathcal{F}_n)|\mathcal{F}_n] = M_n + \mathbb{E}[X_{n+1}|\mathcal{F}_n] - \mathbb{E}[X_{n+1}|\mathcal{F}_n] = 0$$

as M_n is $\mathcal{F}_n$-measurable and $\mathbb{E}[X_{n+1}|\mathcal{F}_n]$ is also $\mathcal{F}_n$-measurable.

b) We note that for $k \leq n$, $\mathbb{E}[X_k|\mathcal{F}_{k-1}]$ and X_{k-1} are $\mathcal{F}_{k-1}$-measurable. Therefore if we sum up until $k = n$, it follows that A_n is $\mathcal{F}_{n-1}$-measurable. But as $\mathcal{F}_{n-1} \subset \mathcal{F}_n$, then it is also $\mathcal{F}_n$-measurable. Finally,

$$A_{n+1} - A_n = \mathbb{E}[X_{n+1}|\mathcal{F}_n] - X_n > 0$$

as $\{X_i\}$ is a sub-martingale.

7. Let $(\Omega, \mathcal{F}, (\mathcal{F}_{t\in T}), P)$ be a filtered probability space. Fix $s, s' \in T$, with $s \leq s'$, as well as $B \in \mathcal{F}_s$. Show that the random time τ defined via $\tau(w) = s$, for $w \in B$, and $\tau(w) = s'$, for $w \in \Omega - B$, is a stopping time.

The result follows trivially from the fact that:

$$\{w \in \Omega : \tau(w) = s\} = \{w \in \Omega : w \in B\} \in \mathcal{F}_s$$
$$\{w \in \Omega : \tau(w) = s'\} = \{w \in \Omega : w \notin B\} \in \mathcal{F}_s$$

where the last equality comes form the fact that a σ-algebra is closed for complements.

8. Let $X = \{X_n, n \in \mathbb{N}\}$ be a martingale on $(\Omega, \mathcal{F}, P)$, and assume that $\mathcal{G} = \{G_n, n \in \mathbb{N}\}$ is a filtration contained in $\mathcal{F}$, and such that X is $\mathcal{G}$-adapted. Then show that X is also a $(\mathcal{G}, P)$-martingale.

As X is a martingale in the larger filtration $\mathcal{F}$, then it is automatically integrable. We need also to assume that X is $\mathcal{G}$-adapted. So the only result that we need to prove concerns the conditional expectation result, i.e., we need to prove that $\mathbb{E}[X_n|\mathcal{G}_{n-1}] = X_{n-1}$. For that note the following:

$$\mathbb{E}[X_n|\mathcal{G}_{n-1}] = \mathbb{E}[\mathbb{E}[X_n|\mathcal{F}_{n-1}]|\mathcal{G}_{n-1}] = \mathbb{E}[X_{n-1}|\mathcal{G}_{n-1}] = X_{n-1}$$

where we have used one of the properties of conditional expectation, the fact that X is a a martingale on $(\Omega, \mathcal{F}, P)$ and, finally, the fact that (we assume that) X is $\mathcal{G}$-adapted.

9. Consider a Poisson process $N = \{N(t), t \geq 0\}$, with $N(t) \sim Poisson(\lambda t)$, with independent and stationary increments. Define $\tilde{N}(t) = N(t) - \lambda t$, for $t \geq 0$. The filtration here is the one generated by N.

a) Show that $\tilde{N}$ as well as $\{\tilde{N}^2(t) - \lambda t, t \geq 0\}$ are both martingales.

b) Using the definition of quadratic variation, show that

$$[N, N]_t = [\tilde{N}, \tilde{N}]_t = N(t)$$

c) Use the previous two parts to show that $\{\tilde{N}_t^2 - [\tilde{N}, \tilde{N}]_t, t \geq 0\}$ is a martingale.

a) Regarding $\tilde{N}$:it is adapted to its own filtratation (as it is a deterministic transformation of the Poisson process, itself adapted to its own filtration), and $\mathbb{E}[|\tilde{N}(t)|] < 2\lambda t < \infty$. Moreover, for $s < t$:

$$\mathbb{E}[\tilde{N}(t)|\mathcal{F}_s^N] = \mathbb{E}[N(t) - \lambda t|\mathcal{F}_s^N] = \mathbb{E}[N(t) - N(s) + N(s) - \lambda t|\mathcal{F}_s^N] = \mathbb{E}[N(t) - N(s)] + N(s) - \lambda t$$
$$= \mathbb{E}[N(t-s)] + N(s) - \lambda t = \lambda(t-s) + N(s) - \lambda t = N(s) - \lambda s = \tilde{N}(s)$$

where in the last equalities we used properties of the Poisson process (namely the fact that it has independent and stationary increments). Also that we conditioned in the filtration generated by N, has it contains the same information as the one generated by $\tilde{N}$.

Regarding $\{\tilde{N}^2(t) - \lambda t, t \geq 0\}$, it is adapted to its own filtration (for the same reasons as the previous process), and $\mathbb{E}[|\tilde{N}^2(t) - \lambda t|] < 2(\lambda + \lambda^2)t < \infty$. Moreover, for $s < t$:

$$\begin{aligned}\mathbb{E}[\tilde{N}^2(t) - \lambda t|\mathcal{F}_s^N] &= \mathbb{E}[(\tilde{N}(s) + N(t) - N(s) - \lambda(t-s))^2|\mathcal{F}_s^N] - \lambda t \\ &= \tilde{N}^2(s) + 2\tilde{N}(s)\mathbb{E}[N(t) - N(s) - \lambda(t-s)] + \\ &+ \mathbb{E}[(N(t) - N(s) - \lambda(t-s))^2] - \lambda t \\ &= \tilde{N}^2(s) + \lambda(t-s) - \lambda t = \tilde{N}^2(s) - \lambda s\end{aligned}$$

where in the last equalities we used properties of the Poisson process (namely the fact that it has independent and stationary increments).

b) Define $\Delta_t^{i,n} = N\left(\frac{ti}{n}\right) - N\left(\frac{t(i-1)}{n}\right)$, for $i = 1, 2, \ldots, n$. Since N is a Poisson process, it follows that $\Delta_t^{i,n} = \mathbb{1}_{\{\tau_k \in (t(i-1)/n, ti/n), \text{ for some } k \in \mathbb{N}\}}$, where (τ_k) denotes the sequence of arrival times for a Poisson process. Note that as $n \to \infty$, $\Delta_t^{i,n} \in \{0, 1\}$ (due to the properties of the Poisson process), and therefore $\Delta_t^{i,n} = (\Delta_t^{i,n})^2$. Thus:

$$[N, N]_t = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta_t^{i,n})^2 = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_t^{i,n} = N(t) - N(0) = N(t).$$

10. Let $\mathcal{G}$ be a σ-algebra defined on the sets $\{\emptyset, A, A^c, \Omega\}$ be a σ-algebra. What is $\mathbb{E}[X|\mathcal{G}]$?

11. Show that if $\{\mathcal{F}_i, i \in J\}$ is any collection of σ-algebras on the same set Ω, then their intersection, $\bigcap_{i\in J} \mathcal{F}_i$, is also a σ-algebra on Ω.

Let $\mathcal{F}$ be a collection of subsets of Ω. $\mathcal{F}$ is said to be a σ-algebra of subsets of Ω if:

(i) There exists at least one $A \subseteq \Omega$ so that $A \in \Omega$.

(ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

(iii) If $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{F}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

Now suppose that $\{\mathcal{F}_i\}_{i\in J}$ are σ-algebras of Ω and define $\mathcal{F} = \bigcap_{i\in J} \mathcal{F}_i$. We shall show that $\mathcal{F}$ is also a σ-algebra of Ω by proving the previous three properties:

(i) For each $i \in J$, there exists a set $A_i \subseteq \Omega$ so that $A_i \in \mathcal{F}_i$. Then, $\bigcap_{i\in J} A_i \subseteq \Omega$ and $\bigcap_{i\in J} A_i \in \mathcal{F}$.

(ii) If $A \in \mathcal{F}$, then $A \in \mathcal{F}_i$ for any $i \in J$. If follows from the second property that $A^c \in \mathcal{F}_i$ for $i \in J$ and consequently $A^c \in \mathcal{F}$.

(iii) If $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{F}$, then $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{F}_i$ for any $i \in J$. By the σ-property, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}_i$ and finally $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

12. Let Σ be a σ-algebra of subsets of X and $Y \subset X$ be a non-empty subset of X. If we denote

$$\Sigma|Y = \{A \cap Y | A \in \Sigma\}$$

then prove that $\Sigma|Y$ is a σ-algebra of subsets of Y.

– As Σ is a σ-algebra, it follows that with $\emptyset = \emptyset \cap Y \in \Sigma|Y$.
– If $B \in \Sigma|Y$, then it exists $A \in \Sigma$ such that $B = A \cap Y$. Since $X - A \in \Sigma$, we get that $Y - B = (X - A) \cap Y \in \Sigma|Y$.
– If $B_1, B_2, \ldots \in \Sigma|Y$, then for each k, $B_k = A_k \cap Y$, for some $A_k \in \Sigma$. Since $\cup_k A_k \in \Sigma$, we find that $\cup_k B_k = (\cup_k A_k) \cap Y \in \Sigma|Y$.

13. Prove that the conditional expectation is linear, i.e., prove that for X and Y such that they are $\mathcal{F}$-adapted,

$$\mathbb{E}[\alpha X + \beta Y|\mathcal{F}] = \alpha\mathbb{E}[X|\mathcal{F}] + \beta\mathbb{E}[Y|\mathcal{F}].$$

By definition of conditional expectation, it follows that for all $A \in \mathcal{F}$, and using the properties of the expectation (a linear operator):

$$\mathbb{E}\left[(\alpha X + \beta Y)\mathbb{1}_A\right] = \mathbb{E}\left[\alpha X\mathbb{1}_A + \beta Y\mathbb{1}_A\right] = \alpha\mathbb{E}\left[X\mathbb{1}_A\right] + \beta\mathbb{E}\left[Y\mathbb{1}_A\right]$$

from which the result follows.

14. Let X be a non-empty set and $\mathcal{A}$ a collection of subsets of X. We call $\mathcal{A}$ an algebra of subsets of X it is is non-empty, closed under complements and such that if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$. Prove that an algebra has the following properties:

a) Contains at least the sets $\emptyset$ and X;
b) It is closed under finite unions, under finite intersections and under set differences (like $A - B$).

a) Take any set $A \in \mathcal{A}$ (there exists at least one since $\mathcal{A}$ is an algebra). Since $\mathcal{A}$ is closed under complements, $A^c \in \mathcal{A}$. Moreover, $X = A \cup A^c \in \mathcal{A}$ and consequently $\emptyset = X^c \in \mathcal{A}$.

b) – $\mathcal{A}$ is closed under finite unions:
We prove this property by induction. If $A_1, A_2 \in \mathcal{A}$, then $A_1 \cup A_2 \in \mathcal{A}$. Now suppose that for any $n \in \mathbb{N}$, if $A_1, ..., A_n \in \mathcal{A}$, then $\bigcup_{k=1}^n A_k \in \mathcal{A}$. It follows that if $A_1, ..., A_n, A_{n+1} \in \mathcal{A}$, then $\bigcup_{k=1}^{n+1} A_k = (\bigcup_{k=1}^n A_k) \cup A_{n+1} \in \mathcal{A}$.
– $\mathcal{A}$ is closed under finite intersections:
We prove that the intersection of two sets is in $\mathcal{A}$. If $A_1, A_2 \in \mathcal{A}$, then $A_1^c, A_2^c \in \mathcal{A}$. Thus, $A_1^c \cup A_2^c \in \mathcal{A}$ and $A_1 \cap A_2 = (A_1^c \cup A_2^c)^c \in \mathcal{A}$. The rest of the proof is by induction and similar to the previous one.
– $\mathcal{A}$ is closed under set differences:
If $A_1, A_2 \in \mathcal{A}$, then $A_1 \setminus A_2 = A_1 \cap A_2^c \in \mathcal{A}$.

15. Let $\{\epsilon_j, j = 1, 2, \cdots\}$ be a sequence of i.i.d. random variables with common distribution

$$P(\epsilon_j = 1) = p; \quad P(\epsilon_j = -1) = q = 1 - p$$

and $\mathcal{F}_n = \sigma(\epsilon_j, 0 \le j \le n), n \ge 0$ their natural filtration. Denote $S_n = \sum_{j=1}^n \epsilon_j$, $n \ge 0$. Prove that $\{M_n = \left(\frac{q}{p}\right)^{S_n}, n \ge 0\}$ is a $\mathcal{F}_n$-martingale.

By construction, S_n is $\mathcal{F}_n$ measurable and therefore so it is M_n, for all n. Also $\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_n}\right] < \infty$ as $S_n \le n$ with probability 1. Then we need to check the martingale equality:

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \mathbb{E}\left[M_n\left(\frac{q}{p}\right)^{\epsilon_{n+1}}\middle|\mathcal{F}_n\right] = M_n\mathbb{E}\left[\left(\frac{q}{p}\right)^{\epsilon_{n+1}}\middle|\mathcal{F}_n\right] = M_n\mathbb{E}\left[\left(\frac{q}{p}\right)^{\epsilon_{n+1}}\right] = M_n\left(p\,\frac{q}{p} + q\,\frac{p}{q}\right) = M_n$$

16. Let X be an integrable random variable, and $\mathcal{H} = \{\mathcal{H}_t, t \ge 0\}$ an increasing family of σ-algebras. Let $X_t = \mathbb{E}[X|\mathcal{H}_t]$. Prove that $\{X_t, t \ge 0\}$ is a martingale with respect to $\mathcal{H}$.

We need to prove the following:

- X_t is $\mathcal{H}_t$ measurable: true, by definition of conditional expectation.
- $\mathbb{E}[|X_t|] < \infty$: Using the definition of X_t, then the Jensen inequality, followed by the Tower property and, finally, using the fact that X is an integrable random variable, it follows that:

$$\mathbb{E}[|X_t|] = \mathbb{E}[|\mathbb{E}[X|\mathcal{H}_t]|] \leq \mathbb{E}[\mathbb{E}[|X||\mathcal{H}_t]] = \mathbb{E}[|X|] < \infty$$

- Assume that $s < t$ and we check the conditional expectation property:

$$\mathbb{E}[X_t|\mathcal{H}_s] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}_t]|\mathcal{H}_f]] = \mathbb{E}[X|\mathcal{H}_s] = X_s$$

where in the above derivation we have used the definition of X_t, the Tower property and again the definition of X_s.

17. Let M and X be two discrete-time processes, such that M is a martingale, and define the martingale transform of M by X as follows:

$$(X.M)_n = \sum_{k=1}^{n} X_k(M_k - M_{k-1})$$

Prove that if X is a bounded, predictable process, then $X.M$ is a martingale.

We need to check the three properties that come in the definition of martingale, which are the following:

- $(X.M)_n$ is $\mathcal{F}_n$ measurable: true, as for $k \leq n$, X_k and M_k are $\mathcal{F}_k$ measurable, M_{k-1} is $\mathcal{F}_{k-1}$ measurable, and therefore their sum and product is $\mathcal{F}_k$ measurable, which means that it is $\mathcal{F}_n$ measurable, as $\mathcal{F}_k \subseteq \mathcal{F}_n, \forall k \leq n$.
- $\mathbb{E}[|(X.M)_n|] < \infty$: this follows as

$$\begin{aligned}
\mathbb{E}[|(X.M)_n|] &\leq \mathbb{E}\left[\sum_{k=1}^{n} |X_k|\,|M_k - M_{k-1}|\right] \leq \mathbb{E}\left[\sum_{k=1}^{n} |X_k|(|M_k| + |M_{k-1}|)\right] \\
&= \sum_{k=1}^{n} \mathbb{E}\left[|X_k|](\mathbb{E}[|M_k|] + \mathbb{E}[|M_{k-1}|])\right] \leq \max_k \mathbb{E}[|X_k|] \sum_{k=1}^{n} (\mathbb{E}[|M_k|] + \mathbb{E}[|M_{k-1}|])] \\
&= K \sum_{k=1}^{n} (\mathbb{E}[|M_k|] + \mathbb{E}[|M_{k-1}|])] < \infty
\end{aligned}$$

where K is such that $X_k \leq K$ with probability one (which exists and it is finite, as X is a bounded process), and $\mathbb{E}[|M_k|] < \infty, \forall k$, as M is a martingale.
- As the process is time-discrete, we need only to prove that $\mathbb{E}[(X.M)_{n+1}|\mathcal{F}_n] = (X.M)_n$, $\forall n \in \mathbb{N}_0$. This follows as:

$$\begin{aligned}
\mathbb{E}[(X.M)_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[(X.M)_n + X_{n+1}\left(M_{n+1} - M_n\right)|\mathcal{F}_n\right]] = (X.M)_n + \mathbb{E}\left[X_{n+1}\left(M_{n+1} - M_n\right)|\mathcal{F}_n\right]] \\
&= (X.M)_n + X_{n+1}\mathbb{E}\left[\left(M_{n+1} - M_n\right)|\mathcal{F}_n\right]] \quad \text{because } X \text{ is predictable} \\
&= (X.M)_n + X_{n+1} \times 0 \quad \text{because } M \text{ is a martingale} \\
&= (X.M)_n
\end{aligned}$$

18. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, P)$ be a filtered probability space. Let $\{M_n, n = 0, 1, 2, ...\}$ be a square integrable martingale.

  a) Show that $\mathbb{E}[M_{n+1}^2|\mathcal{F}_n] \geq M_n^2$, for all n.

  b) Show that $\text{Cov}(M_{n+1} - M_n, M_n) = 0$, for all n.

  c) Let

$$\langle M \rangle_n = \sum_{i=0}^{n-1} \mathbb{E}[(M_{i+1} - M_i)^2|\mathcal{F}_i], \quad \forall n$$

  with $\langle M \rangle_0 = 0$. Argue that $\langle M \rangle_n$ is $\mathcal{F}_n$-measurable, for all n, and that $\{M_n^2 - \langle M \rangle_n, n\}$ is a martingale (without proving the integrability condition).

a) As $\{M_n, n = 0, 1 \ldots\}$ is a martingale, we know that $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$. Then it follows, using the Jensen inequality:

$$\mathbb{E}[M_{n+1}^2|\mathcal{F}_n] \geq (\mathbb{E}[M_{n+1}|\mathcal{F}_n])^2 = M_n^2.$$

b) In order to prove such result, we start by computing $\mathbb{E}[M_{n+1}M_n]$, which we will do using the Tower property (in order to use the martingale condition)

$$\begin{aligned}\mathbb{E}[M_{n+1}M_n] &= \mathbb{E}\left[\mathbb{E}[M_{n+1}M_n|\mathcal{F}_n]\right] = \mathbb{E}\left[M_n\mathbb{E}[M_{n+1}|\mathcal{F}_n]\right] \quad \text{as } M_n \text{ is } \mathcal{F}_n \text{ measurable}\\ &= \mathbb{E}\left[M_n M_n\right] \quad \text{by the martingale property}\\ &= \mathbb{E}[M_n^2]\end{aligned}$$

Therefore, as also $\mathbb{E}[M_n] = \mathbb{E}[M_{n+1}]$, it follows that:

$$\begin{aligned}\mathrm{Cov}(M_{n+1} - M_n, M_n) &= \mathrm{Cov}(M_{n+1}, M_n) - \mathrm{Var}[M_n]\\ &= \mathbb{E}[M_{n+1}M_n] - \mathbb{E}[M_{n+1}]\mathbb{E}[M_n] - \mathbb{E}[M_n^2] + E^2[M_n]\\ &= \mathbb{E}[M_n^2] - E^2[M_n] - \mathbb{E}[M_n^2] + E^2[M_n] = 0.\end{aligned}$$

c) To prove that it is $\mathcal{F}_n$-measurable, we just need to use the properties of conditional expectation, which imply that $\mathbb{E}[(M_{i+1} - M_i)^2|\mathcal{F}_i]$ is $\mathcal{F}_i$ measurable, and therefore it is also $\mathcal{F}_n$ measurable, implying that its sum is also $\mathcal{F}_n$ measurable.
So now we prove the martingale condition, i.e., we prove that $\mathbb{E}[M_{n+1}^2 - \langle M\rangle_{n+1}|\mathcal{F}_n] = M_n^2 - \langle M\rangle_n$.

$$\begin{aligned}\mathbb{E}[M_{n+1}^2 - \langle M\rangle_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[M_{n+1}^2 - \sum_{i=0}^{n}\mathbb{E}\left[(M_{i+1} - M_i)^2|\mathcal{F}_i\right]\Big|\mathcal{F}_n\right]\\ &= \mathbb{E}\left[M_{n+1}^2 - \sum_{i=0}^{n-1}\left(\mathbb{E}\left[(M_{i+1} - M_i)^2|\mathcal{F}_i\right]\right) - \mathbb{E}\left[(M_{n+1} - M_n)^2|\mathcal{F}_n\right]\Big|\mathcal{F}_n\right]\\ &= \mathbb{E}[M_{n+1}^2|\mathcal{F}_n] - \langle M\rangle_n - \mathbb{E}[(M_{n+1} - M_n)^2|\mathcal{F}_n]\\ &= \mathbb{E}[M_{n+1}^2|\mathcal{F}_n] - \langle M\rangle_n - \mathbb{E}[M_{n+1}^2|\mathcal{F}_n] - \mathbb{E}[M_n^2|\mathcal{F}_n] + 2\mathbb{E}[M_nM_{n+1}|\mathcal{F}_n]\\ &= -\langle M\rangle_n - M_n^2 + 2M_n\mathbb{E}[M_{n+1}|\mathcal{F}_n]\\ &= -\langle M\rangle_n - M_n^2 + 2M_nM_n\\ &= M_n^2 - \langle M\rangle_n\end{aligned}$$

which ends the proof.

19. Prove the following properties of conditional expectation:

a) Prove that if $\mathcal{F} = \{\emptyset, \Omega\}$ then $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$, with probability one, where X is a random variable defined on Ω.

b) Let X be an integrable random variable, and $\mathcal{F}_1 \subset \mathcal{F}_2$ be two σ-algebras. Prove that

$$\mathbb{E}[X|\mathcal{F}_2] = \mathbb{E}[X|\mathcal{F}_1]$$

if and only if $\mathbb{E}[X|\mathcal{F}_2]$ is $\mathcal{F}_1$-measurable.

a) Let $Z = \mathbb{E}[X|\mathcal{F}]$. In case $\mathcal{F}$ is the trivial σ-algebra, then in order to have a r.v. measurable with respect to such σ-algebra, it means that it has to be a deterministic one. Therefore it follows that:

$$\mathbb{E}[Z\mathbb{1}_\Omega] = \mathbb{E}[Z\mathbb{1}_\emptyset]$$

So we compute $\mathbb{E}[Z\mathbb{1}_\Omega]$, which is equal to $\mathbb{E}[Z\mathbb{1}_\Omega] = \int_\Omega ZdP = \int_\Omega XdP = \mathbb{E}[X]$ which proves that $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$.

b) Suppose that $\mathbb{E}[X|\mathcal{F}_2]$ is also $\mathcal{F}_1$ measurable. Applying the Tower property, it follows that:

$$\mathbb{E}[X|\mathcal{F}_2] = \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}_2]|\mathcal{F}_1\right] = \mathbb{E}[X|\mathcal{F}_1]$$

which proves that if $\mathbb{E}[X|\mathcal{F}_2]$ is $\mathcal{F}_1$ measurable, then the equality holds.
On the other side, if the equality holds, then it follows that $\mathbb{E}[X|\mathcal{F}_2] = \mathbb{E}[X|\mathcal{F}_1]$ is $\mathcal{F}_1$ measurable, by definition of conditional expectation.

20. Let $\{Y_n, n \in \mathbb{N}\}$ be a sequence of positive independent r.v. with $\mathbb{E}[Y_j] = 1$, for all j. Set

$$X_0 = 1, \quad X_n = \prod_{j=1}^{n} Y_j$$

Show that $\{X_n, n \in \mathbb{N}\}$ is a martingale relative to its natural filtration.

In order to prove that it is a martingale, we need to check the following:

- $\mathbb{E}[|X_n|] < \infty$, which is equivalent to show that $\mathbb{E}[X_n] < \infty$. Then:

$$\mathbb{E}[X_n] = \mathbb{E}\left[\prod_{i=1}^{n} Y_n\right] = \prod_{i=1}^{n} \mathbb{E}[Y_n] = 1 < \infty, \forall n$$

  (because the r.v. are independent)
- X_n is $\mathcal{F}_n$ measurable, as by construction it is always adapted to its natural filtration
- $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n Y_{n+1}|\mathcal{F}_n] = X_n \mathbb{E}[Y_{n+1}|\mathcal{F}_n] = X_n \mathbb{E}[Y_{n+1}] = X_n$ where in the previous equality we have used the fact that X_n is $\mathcal{F}_n$ measurable and that Y_{n+1} is independent of the other random variables.

So the martingale property holds.

21. Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of independent Bernoulli random variables, such that $P(X_i = 0) = 1 - p$ and $P(X_i = 1) = p$. Let $S_n = \sum_{i=1}^{n} X_i$ and, finally, define $Z = \{Z_n, n \in \mathbb{N}\}$, such that:

$$Z_n = \left(\tfrac{1-p}{p}\right)^{2S_n - n}$$

Prove that Z is a martingale.

In order to prove that it is a martingale, we prove the following:

- Z_n depends on S_n, and thus it is obviously adapted to the filtration generated by $\{S_i, i \in \mathbb{N}\}$, which in its turn is adapted to the filtration generated by $\{S_i, i \in \mathbb{N}\}$
- As $S_n \sim Bin(n, p)$, it follows that $S_n \leq n$ and therefore

$$\mathbb{E}[Z_n] \leq \left(\tfrac{1-p}{p}\right)^n < \infty, \forall n$$

- As it is a discrete-time process, we just need to check that $\mathbb{E}[Z_{n+1}|Z_n] = Z_n$:

$$\mathbb{E}[Z_{n+1}|Z_n] = \mathbb{E}\left[Z_n \left(\tfrac{1-p}{p}\right)^{2X_{n+1}-1} \middle| Z_n\right] = Z_n \mathbb{E}\left[\left(\tfrac{1-p}{p}\right)^{2X_{n+1}-1}\right]$$
$$= Z_n \left[(1-p)\left(\tfrac{1-p}{p}\right)^{-1} + p\left(\tfrac{1-p}{p}\right)^{1}\right] = Z_n$$

22. Let X and Y be square integrable random variables and suppose that $\mathbb{E}[X|Y] = Y$ and $\mathbb{E}[Y|X] = X$ a.s. Show that $X = Y$ a.s.

Hint: Compute $\text{Var}(X - Y)$ and using that value, argue the result.

We start by noting that $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[Y]$, which implies that $\mathbb{E}[X-Y]^2 = 0$ and we can simply write $\text{Var}(X-Y) = \mathbb{E}[(X-Y)^2]$. Now if we condition $\mathbb{E}[(X-Y)^2]$ on Y, we get

$$\begin{aligned}\mathbb{E}[(X-Y)^2] &= \mathbb{E}[\mathbb{E}[X^2 - 2XY + Y^2|Y]] = \mathbb{E}[\mathbb{E}[X^2|Y]] - 2\mathbb{E}[Y\mathbb{E}[X|Y]] + \mathbb{E}[Y^2] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[Y^2] + \mathbb{E}[Y^2] = \mathbb{E}[X^2] - \mathbb{E}[Y^2].\end{aligned}$$

Likewise, conditioning $\mathbb{E}[(X-Y)^2]$ on X, we get that $\mathbb{E}[(X-Y)^2] = \mathbb{E}[Y^2] - \mathbb{E}[X^2]$. Thus, since $\mathbb{E}[(X-Y)^2] = -\mathbb{E}[(X-Y)^2]$, we must have $\mathbb{E}[(X-Y)^2] = 0$. We conclude that $\text{Var}(X-Y) = 0$ and consequently $X = Y$ a.s.

23. Let $\{X_n, n\}$ be a sequence of independent Bernoulli r.v. of parameter $\mathbb{E}[X_n] = (1+e)^{-1}$. We define $M_0 = 1$ and for n:

$$M_n = e^{-n+2\sum_{i=1}^n X_i}$$

Prove that $\{M_n, n\}$ is a martingale.

Let $\{\mathcal{F}_n = \sigma(X_n) : n \in \mathbb{N}\}$ be the natural filtration of X_n. In order to prove that $\{M_n : n \in \mathbb{N}\}$ is a martingale adapted to $\{\mathcal{F}_n : n \in \mathbb{N}\}$, we need to prove three properties:

- $M_n \in \mathcal{F}_n$: for $i \leq n$, $X_i \in \mathcal{F}_i \subseteq \mathcal{F}_n$, hence $\sum_{i=1}^n X_i \in \mathcal{F}_n$ and consequently $M_n \in \mathcal{F}_n$.
- M_n is integrable: $0 \leq \mathbb{E}[|M_n|] = \mathbb{E}[M_n] = e^{-n}\mathbb{E}[e^{2\sum_{i=1}^n X_i}] \leq e^{-n}\mathbb{E}[e^{2n}] = e^n < \infty$.
- $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$:

$$\begin{aligned}\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}[e^{-(n+1)+2\sum_{i=1}^{n+1} X_i}|\mathcal{F}_n] = e^{-1}\mathbb{E}[e^{-n+2\sum_{i=1}^n X_i} e^{2X_{n+1}}|\mathcal{F}_n] = e^{-1}M_n\mathbb{E}[e^{2X_{n+1}}|\mathcal{F}_n] \\ &= e^{-1}M_n\mathbb{E}[e^{2X_{n+1}}] = e^{-1}M_n\left(\frac{1}{1+e}e^{2\times 1} + \left(1 - \frac{1}{1+e}\right)e^{2\times 0}\right) = e^{-1}M_n e = M_n.\end{aligned}$$

Obs: In fact, since $\sum_i^n X_i \sim Bin(n, \frac{1}{1+e})$, taking advantage of the moment generating function of the binomial distribution, $\mathbb{E}[e^{2\sum_i^n X_i}] = \left(1 - \frac{1}{1+e} + \frac{1}{1+e}e^2\right)^n = e^n$ and thus $\mathbb{E}[M_n] = 1$.

Chapter 4

Stochastic Calculus

In this chapter, $\{W(t), t \geq 0\}$ denotes a Brownian motion.

1. Our definition of stochastic integral w.r.t. to Brownian motion is based on:

$$\int_0^t g(u)dW(u) := \sum_{k=0}^{n-1} g(t_k)\left[W(t_{k+1}) - W(t_k)\right]$$

where $\{0 = t_0, t_1, \ldots, t_n = t\}$ is one partition of the interval $[0, t]$.

Suppose now that we define a new type of integral, based on the following:

$$\oint_0^t g(u)dW(u) := \sum_{k=0}^{n-1} g(t_{k+1})\left[W(t_{k+1}) - W(t_k)\right]$$

Let $g(t) = W(t_k)$, for $t \in [t_k; t_{k+1}[$.

a) Show that $\mathbb{E}\left[\int_0^t g(u)dW(u)\right] = 0$ but $\mathbb{E}\left[\oint_0^t g(u)dW(u)\right] = t$.

b) Is this new integral still a martingale?

a)

$$\mathbb{E}\left[\int_0^t g(u)dW(u)\right] = \mathbb{E}\left[\sum_{k=0}^{n-1} g(t_k)\left[W(t_{k+1}) - W(t_k)\right]\right] = \sum_{k=0}^{n-1} \mathbb{E}\left[W(t_k)\left[W(t_{k+1}) - W(t_k)\right]\right]$$

Now $(W(t_{k+1}) - W(t_k))$ is independent of $W(t_k)$, and therefore:

$$\sum_{k=0}^{n-1} \mathbb{E}\left[W(t_k)\left[W(t_{k+1}) - W(t_k)\right]\right] = \sum_{k=0}^{n-1} \mathbb{E}\left[W(t_k)\right]\mathbb{E}\left[W(t_{k+1}) - W(t_k)\right] = 0.$$

Similarly,

$$\begin{aligned}
\mathbb{E}\left[\oint_0^t g(u)dW(u)\right] &= \mathbb{E}\left[\sum_{k=0}^{n-1} g(t_{k+1})\left[W(t_{k+1}) - W(t_k)\right]\right] = \sum_{k=0}^{n-1} \mathbb{E}\left[W(t_{k+1})\left[W(t_{k+1}) - W(t_k)\right]\right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}\left[\left[W(t_{k+1}) - W(t_k) + W(t_k)\right]\left[W(t_{k+1}) - W(t_k)\right]\right] \\
&= \sum_{k=0}^{n-1} \mathrm{Var}\left[W(t_{k+1}) - W(t_k)\right] = \sum_{k=0}^{n-1} (t_{k+1} - t_k) = t_n - t_0 = t.
\end{aligned}$$

b) In order to check if this is still a martingale (recall that $\int_0^t g(u)dW(u)$ is a martingale, with g given by $g(t) = W(t_k)$, for $t \in [t_k; t_{k+1}[$), we check first the following conditional condition holds:

$$\mathbb{E}\left[\oint_0^t g(u)dW(u)\Big|\mathcal{F}_s\right] = \oint_0^s g(u)dW(u)$$

Now consider a partition of the time interval $[0,t]$ such that $t_m = s$, for $s < t$, and denote by $\mathcal{F}_s$ the sigma-algebra of the process generated until time s. Then

$$\begin{aligned}
\mathbb{E}\left[\oint_0^t g(u)dW(u)\Big|\mathcal{F}_s\right] &= \mathbb{E}\left[\sum_{k=0}^{n-1} W(t_{k+1})[W(t_{k+1}) - W(t_k)]\Big|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{m-1} W(t_{k+1})\,[W(t_{k+1}) - W(t_k)]\,\Big|\mathcal{F}_s\right] \\
&\quad + \mathbb{E}\left[\sum_{k=m}^{n-1} g(t_{k+1})\,[W(t_{k+1}) - W(t_k)]\,\Big|\mathcal{F}_s\right] \\
&= \oint_0^s g(u)dW(u) + \mathbb{E}\left[\oint_s^t g(u)dW(u)\right] = \oint_0^s g(u)dW(u) + (t-s)
\end{aligned}$$

and therefore this is **not** a martingale!

Note that this means, in particular, that contrary to real integrals (where you can define left or right ladder functions), in the stochastic integral it is crucial the definition of such functions.

2. Let X be a diffusion, and $a : \mathbb{R}_0^+ \to \mathbb{R}^+$ be a deterministic function. Derive the dynamics of aX.

Use Ito's rule, with:

$$f(t,x) = a(t)x; \quad f_t(x,t) = a'(t)x; \quad f_x(t,x) = a(t); \quad f_{xx}(t,x) = 0.$$

Then, assuming that $dX(t) = \mu(t)X(t)dt + \sigma(t)X(t)dW(t)$, it follows that

$$df(t,x) = (a'(t)x + a(t)\mu(t)X(t))dt + a(t)\sigma(t)X(t)dW(t) = a'(t)X(t)dt + a(t)dX(t).$$

Therefore if a is a deterministic function, then the usual derivation rules hold.

3. Regarding the Brownian motion:

   a) Prove that for $0 < t < u$, $\mathrm{Cov}(W(t) - \frac{t}{u}W(u), W(u)) = 0$.

   b) Is $W(t) - \frac{t}{u}W(u)$ independent of $W(u)$? Justify your answer.

   c) Prove that

   $$\mathbb{E}[W(t)|W(u)] = \frac{t}{u}W(u).$$

   Note: this last assertion means that the conditional mean of the Brownian motion is obtained by linearly interpolating between the points we are given.

4. Consider the following stochastic differential equation:

$$dX(t) = -\frac{X(t)}{1-t}dt + dW(t), \quad t \in [0,1[$$

with $X(0) = 0$.

   a) Show that $\mathrm{Cov}(X(s), X(t)) = s(1-t)$, for $0 \leq s \leq t < 1$.

   b) Prove that the solution of such equation is:

   $$X(t) = (1-t)\int_0^t \frac{1}{1-u}dW(u)$$

   for $t \in [0,1[$. What is a reasonable value for $X(1)$?

   c) Derive $\mathbb{E}[X(t)]$. Is X a martingale?

5. Let $W = \{W(t), t \geq 0\}$ be a Brownian motion and $S = \{S(t), t \geq 0\}$, where $S(t) = e^{\mu t + \sigma W(t)}$, with $S(0) = 1$.

   a) Derive an expression for $P(S(t) \leq x)$.

   b) Determine an expression for the conditional expectation $\mathbb{E}[S(t)|\mathcal{F}_s]$, where $s < t$ and $\{\mathcal{F}_s, s \geq 0\}$ is the filtration associated with the process S.

   c) Find conditions on μ and σ under which the process $\{S(t), t \geq 0\}$ is a martingale.

   Note: You may need to use the following fact: the moment generating function of a random variable Y with normal distribution, with parameters μ and σ is given by:

$$\mathbb{E}[e^{sY}] = e^{s\mu + \frac{1}{2}\sigma^2 s^2}, \quad s \in \mathbb{R}.$$

   d) Let $\tau_a = \inf\{t : S(t) = a\}$, defined for $a > 0$. Derive the probability distribution of τ_a.

a)

$$P(S(t) \leq x) = P(e^{\mu t + \sigma W(t)} \leq x) = P\left(W(t) \leq \frac{\ln x - \mu t}{\sigma}\right) = \Phi\left(\frac{\ln x - \mu t}{\sigma\sqrt{t}}\right)$$

for $x \geq 0$.

b) Using the fact that the Brownian motion has stationary and independent increments, we have the following:

$$\mathbb{E}[S(t)|\mathcal{F}_s] = \mathbb{E}[e^{\mu t + \sigma W(t)}|\mathcal{F}_s] = e^{\mu t} e^{\sigma W(s)} \mathbb{E}[e^{\sigma(W(t) - W(s))}|\mathcal{F}_s] = e^{\mu t} e^{\sigma W(s)} \mathbb{E}[e^{\sigma(W(t) - W(s))}]$$
$$= e^{\mu t} e^{\sigma W(s)} \mathbb{E}[e^{\sigma W(t-s)}] = e^{\mu t} e^{\sigma W(s)} e^{\frac{1}{2}(t-s)\sigma^2} = S(s) e^{(\mu + \frac{1}{2}\sigma^2)(t-s)}$$

for $t < s$.

c) As for each t, $S(t)$ is a deterministic function of the Brownian motion, then of course that it is $\mathcal{F}$-adapted. Moreover, $\mathbb{E}[|S(t)|] = \mathbb{E}[e^{\mu t + \sigma W(t)}] < \infty$, for all t, as long as $\mu < \infty$ and $\sigma < \infty$. So we just need to prove the condition on the equality of the conditional expected value:

$$\mathbb{E}[S(t)|\mathcal{F}_s] = e^{\mu t} e^{\sigma W(s)} e^{\frac{1}{2}(t-s)\sigma^2} = S(s) = e^{\mu s + \sigma W(s)}$$

from where we get the following condition:

$$\mu(t - s) + \frac{1}{2}(t - s)\sigma^2 = 0 \iff \mu = -\frac{\sigma^2}{2}.$$

d) In order to solve this exercise we just have to use the following relationship between first passage times of the process W and the process S:

$$\tau_a = \inf\{t : S(t) = a\} = \inf\{t : W(t) = \tfrac{\ln a - \mu t}{\sigma}\} = \tau^W_{\frac{\ln a - \mu t}{\sigma}}$$

Notice that this problem is in fact harder than the usual hitting time problem, as in here we have a moving boundary. But for the time being, we forget this situation and act as if this is a constant boundary. So

$$P(\tau_a < t) = P\left(\tau^W_{\frac{\ln a - \mu t}{\sigma}} < t\right) = 2P(W(t) > \tfrac{\ln a - \mu t}{\sigma}).$$

6. Let x_0 be a real number, a, b and c be sufficiently smooth deterministic functions. Define $X = \{X(t), t \geq 0\}$ as follows:

$$X(t) = a(t)\left(x_0 + \int_0^t b(s)ds + \int_0^t c(s)dW(s)\right)$$

   a) Prove that $X(t)$ is normally distributed for all t.

   b) Show that if $a(t) > 0$ for $t \geq 0$, then X has the following dynamics:

$$dX(t) = \left(a(t)b(t) + \frac{a'(t)}{a(t)}X(t)\right)dt + a(t)c(t)dW(t)$$

c) Now consider the following SDE:

$$dX(t) = -\frac{X(t)}{1-t}dt + dW(t)$$

with $X(0) = 0$ and defined for $t \neq 1$. Prove that the solution to this SDE is the following:

$$X(t) = (1-t)\int_0^t \frac{1}{1-u}dW(u)$$

Is X a martingale?

d) Derive $\mathbb{E}[X(t)]$ and prove that $Cov(X(s), X(t)) = s(1-t)$ for $0 \leq s \leq t < 1$.

7. For a stochastic process $\{X(t), t \geq 0\}$ and for a a partition, we define the quadratic variation process as follows:

$$[X, X]_t = \lim_{\Delta t \to 0} \sum (X(t_{i+1} - X(t_i))^2$$

where $t_i = i\Delta t$ Prove that if X is a Brownian motion, then

$$[X, X]_t = t.$$

8. Assume that X is a continuous martingale, and let α be a constant. Prove that the process $\{Y(t), t \geq 0\}$ defined by:

$$Y(t) = e^{\alpha X(t) - \frac{1}{2}\alpha [X,X]_t}$$

is a martingale.
Hint: Compute $dY(t)$.

9. Use Ito's formula to compute $\mathbb{E}[W^4(t)]$.

10. Let $\{W(t), t \in \mathbb{R}^+\}$ denote the standard Brownian motion. Classify the following processes as martingales or not:

a) $X(t) = 2W(t) - 2$
b) $Y(t) = W^2(t) - t$
c) $Z(t) = t^2W(t) - 2\int_0^t sW(s)ds$

11. Let X and Y be two random variables of the form:

$$X = x_0 + \int_0^T g(s)dW(s)$$
$$Y = y_0 + \int_0^T h(s)dW(s)$$

and g and h having disjoint support, i.e.:

$$g(t)h(t) = 0, \quad P - a.s., 0 \leq t \leq T.$$

Using Itô's formula, prove then that

$$XY = x_0y_0 + \int_0^T [X(s)h(s) + Y(s)g(s)]\, dW(s).$$

12. Let $B = \{B(t) = (1 + \frac{1}{2}W(t))^2\}$ and $Z = \{Z(t) = e^{\frac{1}{2}t}\sin(W(t))\}$. Prove that:

a) Z is a martingale with respect to the filtration of the Brownian motion.
b) B satisfies:

$$dB(t) = \frac{1}{4}dt + \sqrt{B(t)}dW(t)$$

13. Let $X = \{X(t) = W^3(t) - 3tW(t), t \geq 0\}$. Prove, using the definition of martingale, that X is a martingale (note that if X is a r.v. with symmetric density distribution, then $\mathbb{E}[X^3] = 0$).

We note, first, that $X(t) \in \mathcal{F}_t$, trivially, as this is a deterministic function of $W(t)$; then we have to assume that $\mathbb{E}[|X(t)|] < \infty$. So let us prove that the equality of the condition expectation holds:

$$\begin{aligned}
\mathbb{E}[X(t)|\mathcal{F}_s] &= \mathbb{E}[W^3(t) - 3tW(t)|\mathcal{F}_s] = \mathbb{E}[W^3(t)|\mathcal{F}_s] - \mathbb{E}[3tW(t)|\mathcal{F}_s] = \mathbb{E}[(W(t) - W(s) + W(s))^3|\mathcal{F}_s] - 3tW(s) \\
&= \mathbb{E}[(W(t) - W(s))^3 + 3(W(t) - W(s))^2W(s) + 3(W(t) - W(s))W^2(s)|\mathcal{F}_s] + W^3(s) - 3tW(s) \\
&= \mathbb{E}[(W(t) - W(s))^3] + 3W(s)\mathbb{E}[(W(t) - W(s))^2] + 3W^2(s)\mathbb{E}[(W(t) - W(s))] - 3tW(s) + W^3(s) \\
&= 0 + 3W(s)(t - s) - 3tW(s) + W^3(s) = W^3(s) - 3sW(s) = X(s)
\end{aligned}$$

14. Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, such that the Ito's formula may be applied.

a) Show that $f(t, W(t))$ is a martingale if and only if

$$\frac{\delta}{\delta t} f(t,x) + \frac{1}{2}\frac{\delta^2}{\delta x^2} f(t,x) = 0, \quad \forall t, x$$

b) As an application, show that $\{W^2(t) - t, t \geq 0\}$ is a martingale. Show also this result from the definition of a martingale (the equality of the conditional expectation).

c) For $a, b \in \mathbb{R}$, define the process X by $\{X(t) = e^{aW(t)-bt}, t \geq 0\}$. Determine the relationship between a and b in order for X to be a martingale.

a) Assume the necessary conditions, in order to apply Ito's lemma. Then if we apply it to the function f, we derive the following SDE:

$$df(t,W(t)) = \frac{\delta}{\delta t} f(t,W(t))dt + \frac{\delta}{\delta x} f(t,W(t))dW(t) + \frac{1}{2}\frac{\delta^2}{\delta x^2} f(t,W(t))(dW(t))^2$$

i.e.,

$$df(t,W(t)) = \left(\frac{\delta}{\delta t} f(t,W(t)) + \frac{1}{2}\frac{\delta^2}{\delta x^2} f(t,W(t))\right) dt + \frac{\delta}{\delta x} f(t,W(t))dW(t)$$

as $(dW(t))^2 = dt$. Thus f is a diffusion process with drift $\frac{\delta}{\delta t} f(t,W(t)) + \frac{1}{2}\frac{\delta^2}{\delta x^2} f(t,W(t))$ and therefore it is a martingale iff its drift is zero, giving the result (as it has to hold for all $W(t) = x \in \mathbb{R}$).

b) In this case the function f is the following: $f(t,x) = x^2 - t$, which is, at least, twice differentiable for all variables. Besides:

$$\frac{\delta}{\delta t} f(t,x) + \frac{1}{2}\frac{\delta^2}{\delta x^2} f(t,x) = -1 + 1 = 0.$$

If we use the definition of martingale, then it follows that X is $\mathcal{F}$ adapted (trivial, as it is a deterministic function of $W(t)$, for all t) and

$$\mathbb{E}[|X(t)|] \leq \mathbb{E}[W^2(t)] + t < 2t < \infty.$$

So, regarding the conditional expectation:

$$\begin{aligned}
\mathbb{E}[W^2(t) - t|\mathcal{F}_s] &= \mathbb{E}[(W(t) - W(s) + W(s))^2|\mathcal{F}_s] - t \\
&= \mathbb{E}[(W(t) - W(s))^2 + W^2(s) + 2(W(t) - W(s))W(s)|\mathcal{F}_s] - t \\
&= (t - s) + W^2(s) - t = W^2(s) - s = X(s).
\end{aligned}$$

c) Set $f(t,x) = e^{ax-bt}$, so that $f_t = -bf(t,x)$, $f_x = af(t,x)$ and $f_{xx} = a^2 f(t,x)$. Then X is a martingale if and only if:

$$-bf(t,x) + \frac{1}{2}a^2 f(t,x) = 0 \Leftrightarrow -b + \frac{1}{2}a^2 = 0.$$

15. We define the pathwise covariation of X and Y over $[0,t]$ to be:

$$[X,Y]_t = \lim_{n\to\infty} \sum_{i=1}^{n} (X(t\,\tfrac{i}{n}) - X(t\,\tfrac{i-1}{n}))(Y(t\,\tfrac{i}{n}) - Y(t\,\tfrac{i-1}{n}))$$

provided the previous limit exists. If $X = Y$ then we call it the quadratic variation of X over $[0,t]$.

a) Prove that for the Brownian motion, the following holds:

$$[W,W]_t = t, \forall t$$

Probably you will need to use the strong law of large numbers, that states that $\bar{X}_n \xrightarrow{\text{a.s.}} \mathbb{E}[X]$ (i.e., the sample average converges almost surely to the expected value)

b) Let W and $\tilde{W}$ be two independent Brownian motions. Show that in this case:

$$[\tilde{W}, W]_t = 0, \forall t$$

a) From the properties of the Brownian motion, it follows that $W(t\frac{i}{n}) - W(t\frac{i-1}{n}) \sim N(0, \frac{t}{n})$. Thus

$$[W, W]_t = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{\frac{t}{t}} Z_i\right)\left(\sqrt{\frac{t}{t}} Z_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{\frac{t}{n}} Z_i\right)^2 = \lim_{n\to\infty} \frac{t}{n} \sum_{i=1}^{n} (Z_i)^2 = t$$

with $Z_i \sim Z \sim N(0,1)$, where the last equality follows by the strong law of large numbers, as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (Z_i)^2 = \mathbb{E}[Z^2] = 1$$

(the equalities have to be read in the almost sure sense).

b) The reasoning is similar, except that in this case we end up with

$$[W, \tilde{W}]_t = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{\frac{t}{t}} Z_i\right)\left(\sqrt{\frac{t}{t}} Y_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{\frac{t}{n}} Z_i Y_i\right) = \lim_{n\to\infty} \frac{t}{n} \sum_{i=1}^{n} (Z_i Y_i) = 0$$

with $Z_i, Y_i \sim Z \sim N(0,1)$, independent, and also from the fact that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (Z_i Y_i)^2 = E^2[Z] = 0$$

by the independence assumption.

16. Fix T and consider the following SDE:

$$dX(t) = \frac{y - X(t)}{T - t} dt + \sigma dW(t), \quad t \in [0, T)$$

with $X(0) = x \in \mathbb{R}$.

a) Is $X = \{X(t), t \geq 0\}$ a martingale?

b) Define a new process $R = \{R(t), t \geq 0\}$ such that $R(t) = \frac{X(t)-y}{T-t}$ for $t \in [0, T)$. Show that

$$dR(t) = \frac{\sigma}{T - t} dW(t)$$

c) Prove that

$$X(t) = y + (T - t)R(t) = (1 - \tfrac{t}{T})x + \frac{t}{T} y + \sigma(T - t) \int_0^t \frac{1}{T - u} dW(u)$$

d) Argue that then X is a Gaussian process, and compute its expected value and covariance-function.

a) X is a diffusion process, with drift term given by $\frac{y-X(t)}{T-t}$, which is different from zero with probability one. Thus it is not a martingale.

b) We apply Ito's formula to this transformation, with $f(t, x) = \frac{x-y}{T-t}$. Thus $f_t(t, x) = \frac{x-y}{(T-t)^2}$, $f_x(t, x) = \frac{1}{T-t}$ and $f_{xx}(t, x) = 0$. So

$$\begin{aligned} dR(t) &= \left(\frac{X(t) - y}{(T - t)^2} + \frac{y - X(t)}{T - t} \times \frac{1}{T - t} + 0\right) dt + \sigma \frac{1}{T - t} dW(t) \\ &= \frac{\sigma}{T - t} dW(t). \end{aligned}$$

c) In view of the last exercise, it follows that

$$R(t) = R(0) + \int_0^t \frac{\sigma}{T-u} dW(u)$$

On the other hand, $X(t) = (T-t)R(t) + y$. Therefore

$$X(t) = (T-t)R(0) + \sigma(T-t) \int_0^t \frac{1}{T-u} dW(u) + y$$

As $X(0) = TR(0) = x$, it follows that $R(0) = x/T$. Thus

$$X(t) = (T-t)x/T + \sigma(T-t) \int_0^t \frac{1}{T-u} dW(u) + y.$$

d) For each t, $X(t)$ is a sum of a deterministic term $((T-t)x/T)$ with an Ito's integral which is known to have a normal distribution. Thus X is indeed a Gaussian process. Regarding its expected value:

$$\begin{aligned}
\mathbb{E}[X(t)] &= \mathbb{E}[(T-t)x/T + \sigma(T-t) \int_0^t \frac{1}{T-u} dW(u) + y] \\
&= (T-t)x/T + y + \sigma(T-t)\mathbb{E}\left[\int_0^t \frac{1}{T-u} dW(u)\right] \\
&= (T-t)x/T + y
\end{aligned}$$

in view of the Ito's isometry. The covariance-function: assume that $s < t$, so that

$$\begin{aligned}
\mathrm{Cov}(X(t), X(s)) &= \mathrm{Cov}\Big((T-t)x/T + \sigma(T-t) \int_0^t \frac{1}{T-u} dW(u) + y, \\
&\qquad (T-s)x/T + \sigma(T-s) \int_0^s \frac{1}{T-u} dW(u) + y\Big) \\
&= \mathrm{Cov}\left(\sigma(T-t) \int_0^t \frac{1}{T-u} dW(u), \sigma(T-s) \int_0^s \frac{1}{T-u} dW(u)\right) \\
&= \sigma^2(T-t)(T-s)\Bigg\{\mathrm{Var}\left(\int_0^s \frac{1}{T-u} dW(u)\right) \\
&\quad + \mathrm{Cov}\left[\int_s^t \frac{1}{T-u} dW(u), \int_0^s \frac{1}{T-u} dW(u)\right]\Bigg\} \\
&= \sigma^2(T-t)(T-s) \int_0^s \frac{1}{(T-u)^2} du + 0 = \sigma^2(T-t)(T-s)\left(\frac{1}{T-s} - \frac{1}{T}\right) \\
&= \sigma^2(T-t)\frac{s}{T}.
\end{aligned}$$

17. Let $X = \{X(t), t \geq 0\}$ and $Y = \{Y(t), t \geq 0\}$ be two stochastic processes satisfying the following system of SDE's:

$$\begin{aligned}
dX(t) &= \alpha X(t)dt + Y(t)dW(t) \\
dY(t) &= \alpha Y(t)dt - X(t)dW(t)
\end{aligned}$$

with $X(0) = x_0$ and $Y(0) = y_0$, deterministic constants.

a) Show that $R = \{R(t), t \geq 0\}$, with $R(t) = X^2(t) + Y^2(t)$, is deterministic.

b) Compute $\mathbb{E}[X(t)]$ and $\mathrm{Cov}(X(t), Y(t))$ and determine for which value of α the process R becomes a deterministic constant.

a) Compute first $dX^2(t)$ and $dY^2(t)$, using Ito's formula, with $f(t,x) = x^2$, $f_x(t,x) = 2x$ and $f_{xx}(t,x) = 2$:

$$dX^2(t) = (2\alpha X^2(t) + \frac{1}{2}2Y^2(t))dt + 2X(t)Y(t)dW(t)$$
$$dY^2(t) = (2\alpha Y^2(t) + \frac{1}{2}2X^2(t))dt - 2X(t)Y(t)dW(t)$$

Therefore

$$dR(t) = (2\alpha(X^2(t) + Y^2(t)) + X^2(t) + Y^2(t))dt + 0dW(t) = (2\alpha + 1)R(t)dt$$

Therefore R has zero volatility, and thus it is a deterministic process.

b) The SDE for the process X can also be written as follows, in integral way:

$$X(t) = X(0) + \alpha \int_0^t X(s)ds + \int_0^t Y(s)dW(s)$$

Applying the expect value operator, one gets the following:

$$\mathbb{E}[X(t)] = X(0) + \alpha \int_0^t \mathbb{E}[X(s)]ds$$

(in view of the Ito's isometry). Let $f(t) = \mathbb{E}[X(t)]$; then if we differentiate the previous equation, we end up with the following ODE:

$$f'(t) = \alpha f(t)$$

whose solution is $f(t) = f(0) + e^{\alpha t}$, i.e.,

$$\mathbb{E}[X(t)] = X(0) + e^{\alpha t}.$$

18. For $a > 0$, define a the process $\tilde{W} = \{\tilde{W}(t), t \geq 0\}$, where $\tilde{W}(t) = \frac{1}{\sqrt{a}}W(at)$. Show that $\tilde{W}$ is also a Brownian motion.

In order to prove that it is a Brownian motion, we need to prove the following assertions:

- $\tilde{W}(0) = aW(0) = 0$
- It has continuous sample paths (as W also has continuous sample paths)
- If $s < t$:

$$\begin{aligned}\tilde{W}(t) - \tilde{W}(s) &= \frac{1}{\sqrt{a}}(W(at) - W(as)) \\ &\sim \mathcal{N}\left(\frac{1}{\sqrt{a}}, \left(\frac{1}{\sqrt{a}}\right)^2 (at - as)\right) \\ &\sim \mathcal{N}(0, t - s)\end{aligned}$$

  in view of the properties of the normal distribution.
- If $u < s < t$, then $au < as < at$, and therefore $W(at) - W(as)$ is independent of $W(au)$. Multiplying by $\frac{1}{\sqrt{a}}$ won't change this.

Thus $\tilde{W}$ is indeed a Brownian motion.

19. Let W and W^* be two independent Brownian motions, and define the processes $X = \{X(t), t \geq 0\}$ and $Y = \{Y(t), t \geq 0\}$ as follows:

$$X(t) = x + \mu_X t + \sigma_X W(t), \quad Y(t) = y + \mu_Y t + \sigma_Y(\rho W(t) + \sqrt{1-\rho^2}W^*(t))$$

with $\sigma_X(\sigma_Y) > 0$ and $|\rho| \leq 1$.

a) Show that X is a Brownian motion but not a standard one. In particular, derive the two first order moments of $X(t)$. Is X a martingale?

b) Show that $\{\rho W(t) + \sqrt{1-\rho^2}W^*(t), t \geq 0\}$ is a Brownian motion.

c) What is the relationship between X and Y when $\rho = 0$? And when $\rho = 1$ or $\rho = -1$?

d) Show that the path-wise covariation of X and Y over $[0,t]$ is given by

$$[X,Y]_t = \lim_{n\to\infty}\sum_{i=1}^{n}\left(X(t\,\tfrac{i}{n}) - X(t\,\tfrac{i-1}{n})\right)\left(Y(t\,\tfrac{i}{n}) - Y(t\,\tfrac{i-1}{n})\right) = \rho\sigma_X\sigma_Y t.$$

a) As $\mu_X t$ is a continuous function in t, it follows that $\{X(t)\}$ has continuous sample paths. Moreover, the increments are independent, in view of independent increments of the Brownian motion. Also, as $X(t+s) - X(t) = \mu_X s + \sigma_X(W(t+s) - W(t)) \overset{d}{=} \mu_X s + W(s)$, does not depend on t. Note that this property is not exactly the way we define stationary increments, and so we will take a more general definition of stationary process. In fact although $X(t+s) - X(t) \neq X(s)$, still $X(t+s+z) - X(t+z) = X(t+s) - X(t)$, for all t, s, z, where the equalities should be read in distribution sense. Also $X(0) = x$ and $X(t) \sim N(x + \mu_X t + \sigma_X^2 t)$, due to the properties of the normal distribution, and therefore X is a non-standard Brownian motion.

b) It follows directly from the properties of Brownian motion and the fact that a linear combination of independent normal variables is still a normal distribution. Also note that $\mathbb{E}[\rho W(t) + \sqrt{1-\rho^2}W^*(t)] = 0$ and $\mathrm{Var}[\rho W(t) + \sqrt{1-\rho^2}W^*(t)] = \rho^2 t + (1-\rho^2)t = t$.

c) If $\rho = 0$ then $Y(t) = y + \mu_Y t + \sigma_Y W^*(t))$ and therefore X and Y are independent, as W and $W^\star$ are also independent. If $\rho = 1$ then $Y(t) = y + \mu_Y t + \sigma_Y W(t))$ and therefore X and Y are perfectly correlated, also in sample path, meaning that for all $w \in \Omega$, $Y(t)(w) - y - \mu_Y t = X(t)(w) - x - \mu_X t$. If $\rho = -1$, it is similar to the last case, except that $Y(t) = y + \mu_Y t - \sigma_Y W(t))$ meaning that $Y(t)(w) - y - \mu_Y t = -X(t)(w) + x + \mu_X t$.

d) Expand the formula:

$$\begin{aligned}
[X,Y]_t &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(X(t\,\tfrac{i}{n}) - X(t\,\tfrac{i-1}{n})\right)\left(Y(t\,\tfrac{i}{n}) - Y(t\,\tfrac{i-1}{n})\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\mu_X\tfrac{t}{n} + \sigma_X(W(t\,\tfrac{i}{n}) - W(t\,\tfrac{i-1}{n}))\right)\\
&= \left(\mu_Y\tfrac{t}{n} + \sigma_Y\rho(W(t\,\tfrac{i}{n}) - W(t\,\tfrac{i-1}{n})) + \sigma_Y\sqrt{1-\rho^2}(W^*(t\,\tfrac{i}{n}) - W^*(t\,\tfrac{i-1}{n}))\right)\\
&\overset{d}{=} \lim_{n\to\infty}\sum_{i=1}^{n}\left(\mu_X\tfrac{t}{n} + \sigma_X W(\tfrac{t}{n})\right)\left(\mu_Y\tfrac{t}{n} + \sigma_Y\rho W(\tfrac{t}{n}) + \sigma_Y\sqrt{1-\rho^2}W^*(\tfrac{t}{n})\right)\\
&= \lim_{n\to\infty}\frac{nt\mu_X\mu_Y}{n^2} + \sigma_X\sigma_Y\rho\lim_{n\to\infty}W(\tfrac{t}{n}) && (\dagger)\\
&= \sigma_X\sigma_Y\rho\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} nW^2(\tfrac{t}{n})\\
&= \sigma_X\sigma_Y\rho t
\end{aligned}$$

where in the fifth equality (†), we used the fact that W and W^* are independent, and with zero mean, and in the last equation we used the strong law of large numbers, a long with the fact that

$$\mathbb{E}[nW^2(\tfrac{t}{n})] = n\frac{t}{n} = t.$$

20. Consider the processes $\{X(t) : t \geq 0\}$ and $B(t) : t \geq 0\}$, where

$$\begin{aligned} X(t) &= W(t) - tW(1)\\ B(t) &= (t+1)X(\tfrac{t}{t+1}) \end{aligned}$$

Show that $\{B(t), t \geq 0\}$ is also a Brownian motion.

21. Let X and Y be two Ito processes, with respect to the same Brownian motion W.

a) Show that $X + Y = \{X(t) + Y(t), t \geq 0\}$ is also an Ito process (using their integral representations).

b) Prove that

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t).$$

hint: $X(t)Y(t) = \frac{1}{2}((X+Y)^2(t) - X^2(t) - Y^2(t))$

a) Consider the following integral representations of the stochastic processes:

$$X(t) = x_0 + \int_0^t \mu_X(s, X(s))ds + \int_0^t \sigma_X(s, X(s))dW(s)$$
$$Y(t) = y_0 + \int_0^t \mu_Y(s, Y(s))ds + \int_0^t \sigma_Y(s, Y(s))dW(s)$$

(note that we are assuming here that X and Y are driven by the same Brownian motion, in the sample-path meaning. What happens if this is not true?) where μ_X, μ_Y, σ_X and σ_Y are stochastic processes with some regular properties (namely, they are $\mathcal{F}^W$-adapted). Then if you sum $X(t)$ and $Y(t)$, for all $t \geq 0$:

$$(X+Y)(t) = X(t) + Y(t) = (x_0 + y_0) + \int_0^t (\mu_X(s, X(s)) + \mu_Y(s, Y(s)))ds$$
$$+ \int_0^t (\sigma_X(s, X(s)) + \sigma_Y(s, Y(s)))dW(s)$$

where, in view of the properties of μ_X and μ_Y, $\mu_X + \mu_y$ has the necessary regularity conditions (in particular, it is $\mathcal{F}^W$-adapted). Moreover, the previous result holds because both the Riemann and the Ito's integrals are additive. (If you wish, prove that the Ito integral is additive, using the definition of Ito integrals).

b) Using the Ito's formula: $d(X^2(t)) = 2X(t)dX(t) + (dX(t))^2$ (prove it; trivial exercise) and similarly for Y. Moreover $X + Y$ is also an Ito process, and

$$\begin{aligned} d(X(t) + Y(t)) = d\bigg(&x_0 + \int_0^t \mu_X(s, X(s))ds + \int_0^t \sigma_X(s, X(s))dW(s) + \\ &y_0 + \int_0^t \mu_Y(s, Y(s))ds + \int_0^t \sigma_Y(s, Y(s))dW(s)\bigg) \\ &= \mu(t, X(t))dt + \sigma(t, X(t))dW(t) + \mu_Y(t, Y(t))dt + \sigma_Y(t, Y(t))dW(t) \\ &= dX(t) + dY(t) \end{aligned}$$

Therefore, making $Z(t) = X(t) + Y(t)$, we have

$$\begin{aligned} dZ^2(t) &= 2Z(t)dZ(t) + (dZ(t))^2 = 2(X(t) + Y(t))(dX(t) + dY(t)) + (dX(t) + dY(t))^2 \\ &= 2X(t)dX(t) + 2Y(t)dY(t) + 2X(t)dY(t) + 2Y(t)dX(t) + d^2X(t) + d^2Y(t) + 2dX(t)dY(t) \end{aligned}$$

Therefore, using the expression found for $dZ^2(t)$,

$$\begin{aligned} \frac{1}{2}d\left((X(t) + Y(t))^2 - X^2(t) - Y^2(t)\right) &= \frac{1}{2}dZ^2(t) - \frac{1}{2}dX^2(t) - \frac{1}{2}dY^2(t) \\ &= \frac{1}{2}dZ^2(t) - \frac{1}{2}\left(2X(t)dX(t) - d^2X(t)\right) - \frac{1}{2}\left(2Y(t)dY(t) - d^2Y(t)\right) \\ &= X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t). \end{aligned}$$

22. Let $M = \{M(t), t \geq 0\}$, with

$$M(t) = \frac{1}{\sqrt{1-t}} \exp\left(-\frac{W^2(t)}{2(1-t)}\right)$$

a) Derive a stochastic differential equation such that M is a solution.

b) Show that M is a martingale.

c) Compute $\mathbb{E}[M(t)]$.

23. Consider the process $X = \{X(t), t \geq 0\}$, such that

$$dX(t) = a(t)dt + \sigma(t)dW(t)$$

where $a = \{a(t), t \geq 0\}$ and $\sigma = \{\sigma(t), t \geq 0\}$ are processes adapted to the filtration generated by W and such that $\mathbb{E}[\int_0^\infty X^2(s)ds] < \infty$.

a) Show, using the definition of martingale and properties of the Ito's integral, that X will not be a martingale with respect to the filtration generated by W unless $a(t) = 0$, $\forall t$.

b) For a certain function h, define the following process: $M = \{M(t), t \geq 0\}$ with

$$M(t) = \exp\left(-\int_0^t h(s)dW(s) - \frac{1}{2}\int_0^t h^2(s)ds\right)$$

Prove that $dM(t) = -h(t)M(t)dW(t)$. Is M a martingale?

c) Choosing $h(t) = a(t)/\sigma(t)$, prove that $\{X(t)M(t), t \geq 0\}$ is also a martingale. Comment this result.

a) X is an Ito process, its integral representation is:

$$X(t) = x_0 + \int_0^t a(y)dy + \int_0^t \sigma(y)dW(y).$$

Then, assuming the necessary conditions for a and σ:

$$\mathbb{E}[X(t)|\mathcal{F}_s] = \mathbb{E}[X(s)] + \mathbb{E}\left[\int_s^t a(y)dy\right] + \mathbb{E}\left[\int_s^t \sigma(y)dW(y)\right] = X(s) + \int_s^t \mathbb{E}[a(y)]dy$$

(using the properties of the Brownian motion, namely independent increments, and the first Ito isometry). Thus X will only be a martingale if and only if $\int_s^t \mathbb{E}[a(y)]dy = \int_s^t \int_\Omega a(y,w)dF_y(w)dy = 0, \forall s < t$, where F is the distribution function of $a(y)$. But using the fundamental theorem of calculus, this means that $a(y,w) = 0, \forall w \in \Omega, y \in \mathbb{R}^+$, and thus we prove the result.

b) Let $g(t, w(t)) = e^{-\int_0^t h(s)dW(s) - \frac{1}{2}\int_0^t h^2(s)ds}$ and apply Ito's rule in order to derive a SDE for M. Note that this function is continuous in t (the first component) and in the second component (the Ito integral is normally distributed). Then

$$\begin{aligned} g_t(t, W(t)) &= -\frac{1}{2}h^2(t)M(t) \\ g_x(t, W(t)) &= -h(t)M(t) \\ g_{xx}(t, W(t)) &= h^2(t)M(t) \end{aligned}$$

where we use the short-hand notation for the partial derivatives. Then, from Ito's formula, it follows that:

$$dM(t) = \{-\frac{1}{2}h^2(t)M(t) + 0 + \frac{1}{2}h^2(t)M(t)\}dt - h(t)M(t)dW(t) = -h(t)dW(t)$$

So M is also an Ito process, with drift equal to zero, and thus it is a martingale.

c) We use the formula (proved in the beginning of this test) for product of two diffusions (depending on the same Brownian motion):

$$\begin{aligned} d(X(t)M(t)) &= X(t)dM(t) + M(t)dX(t) + (dX(t))(dM(t)) \\ &= -X(t)\frac{a(t)}{\sigma(t)}M(t)dW(t) + M(t)(a(t)dt + \sigma(t)dW(t)) \\ &\quad + (a(t)dt + \sigma(t)dW(t))\left(-\frac{a(t)}{\sigma(t)}M(t)dW(t)\right) \\ &= \left(\sigma(t) - \frac{a(t)}{\sigma(t)}X(t)\right)M(t)dW(t) \end{aligned}$$

where in the last derivation we used the multiplication table $dtdW(t) = 0$ and $(dW(t))^2 = dt$. So the process $\{X(s)M(s), s \geq 0\}$ is a diffusion. If $\mathbb{E}[X^2(t)M^2(t)] < \infty$, then it is a martingale, as the drift coefficient is zero.

Then, from what we know from Girsanov theorem, it means that M is the Radon-Nikodym of Q The risk neutral measure) w.r.t. P (the standard measure of the Brownian motion W). Note also that in fact we can derive this measure Q: it is the measure of the process $\tilde{W}$ such that:

$$d\tilde{W}(t) = \frac{a(t)}{\sigma(t)}dt + dW(t)$$

Then:

$$dX(t) = a(t)dt + \sigma(t)dW(t) = a(t)dt + \sigma(t)\left(-\frac{a(t)}{\sigma(t)}dt + d\tilde{W}(t)\right) = \sigma(t)d\tilde{W}(t).$$

24. Let $\tilde{W} = \{\tilde{W}(t) = a^{-1/2}W(at), t \geq 0\}$, with $a > 0$.

a) Prove that $\tilde{W}$ is still a Brownian motion.

b) Let $\varepsilon = \mathbb{1}_{\{\tilde{W}(T)>0\}}$ and define the process M via $M(t) = \mathbb{E}[\varepsilon|\tilde{\mathcal{F}}_t]$, where $\tilde{\mathcal{F}}$ is the filtration generated by $\tilde{W}$. Compute M. Is M a martingale?

a) The following properties hold, in view of the properties of the original Brownian motion W:

- $\tilde{W}(0) = a^{-1/2}W(0) = 0$
- Stationary increments: let $t_1, t_2, \ldots, t_k, s \geq 0$ and $k \in \mathbb{N}$, all arbitrary. Then

$$\begin{aligned}
&(\tilde{W}(t_1 + s) - \tilde{W}(s), \tilde{W}(t_2 + s) - \tilde{W}(s), \ldots, \tilde{W}(t_k + s)) - \tilde{W}(s)) \\
&= (a^{-1/2}(W(a(t_1 + s)) - a^{-1/2}W(as)), a^{-1/2}(W(a(t_2 + s)) - a^{-1/2}W(as)), \ldots, \\
&\quad a^{-1/2}(W(a(t_k + s)) - a^{-1/2}W(as))) \\
&= a^{-1/2}((W(a(t_1 + s)) - W(as)), (W(a(t_2 + s)) - W(as)), \ldots, (W(a(t_k + s)) - W(as))) \\
&\overset{d}{=} a^{-1/2}(W(at_1), W(at_2), \ldots, W(at_k)) \\
&= (\tilde{W}(t_1), \tilde{W}(t_2), \ldots, \tilde{W}(t_k))
\end{aligned}$$

  where the $\overset{d}{=}$ holds in view of the stationary increments for the Brownian motion.
  A similar argument is used to prove independent increments, once again writing $\tilde{W}$ with the help of W and using the properties of the Brownian motion.
- Normal distribution

$$P(\tilde{W}(t) \leq x) = P(a^{-1/2}W(at) \leq x) = P(W(at) \leq xa^{1/2}) = \Phi\left(\frac{xa^{1/2}}{\sqrt{at}}\right) = \Phi\left(\frac{x}{\sqrt{t}}\right)$$

  which proves that $\tilde{W}(t) \sim N(0, t)$.
- The continuity of the sample paths follows from the fact that

$$\lim_{t \to t_0^+} \tilde{W}(t) = a^{-1/2} \lim_{t \to t_0^+} W(at) = a^{-1/2}W(t_0)$$

  in view of the continuity of the Brownian motion.

Therefore we proved that $\tilde{W}$ is indeed a Brownian motion, and that $\{\tilde{W}(t), t \geq 0\} \overset{d}{=} \{W(t), t \geq 0\}$.

b) First of all, as $\tilde{W}$ is also a Brownian motion, we can prove this result for $\varepsilon = \mathbb{1}_{\{W(T)>0\}}$ and M defined, accordingly, as $M(t) = \mathbb{E}[\varepsilon|\mathcal{F}_t]$, where $\mathcal{F}$ is the martingale generated by the Brownian motion W. Then, for $t > 0$:

$$\begin{aligned}
M(t) &= \mathbb{E}[\mathbb{1}_{\{W(T)>0\}}|\mathcal{F}_t] = P(W(T) > 0|\mathcal{F}_t) = P(W(T) - W(t) + W(t) > 0|\mathcal{F}_t) \\
&= P(W(T) - W(t) > -W(t)) = P(Z > -W(t)) = 1 - \Phi\left(\frac{W(t)}{\sqrt{T-t}}\right)
\end{aligned}$$

where $Z \sim N(0, T - t)$ and is independent of $\mathcal{F}_t$, in view of the properties of the Brownian motion (independent increments). So $M = \{M(t) = 1 - \Phi(\frac{W(t)}{\sqrt{T-t}}), t \geq 0\}$. Regarding the martingale property:

- The process M is clearly $\mathcal{F}$-adapted: $1 - \Phi(\frac{W(t)}{\sqrt{T-t}}) \in \mathcal{F}_t$.
- $\mathbb{E}[|M(t)|] = \mathbb{E}[M(t)] \in (0, 1)$, in view of the definition of the indicator function $\mathbb{1}$.
- For $t > s$, arbitrary, it follows that $\mathcal{F}_s \in \mathcal{F}_t$. Moreover, $\mathcal{F}_t^M = \mathcal{F}_t$, $\forall t$, by definition of $M(t)$, where $\mathcal{F}^M$ denotes the filtration generated by M. Therefore, from the properties of conditional expectation:

$$\mathbb{E}[M(t)|\mathcal{F}_s^M] = \mathbb{E}\left[\mathbb{E}[\varepsilon|\mathcal{F}_t]|\mathcal{F}_s\right] = \mathbb{E}[\epsilon|\mathcal{F}_s] = M(s).$$

25. Let $Z = \{Z(t), t \geq 0\}$, where:

$$Z(t) = (1 - t)\int_0^t \frac{dW(s)}{1 - s}$$

a) Compute the expected value of $Z(t)$ and its covariance function $\text{Cov}(Z(t), Z(s))$.

b) Show that:

$$Z(t) = W(t) - (1-t)\int_0^t \frac{W(s)}{(1-s)^2} ds$$

26. Define $Y(t) = W(t) - tW(1)$, for $t \in [0,1]$.

a) Derive the moment generator function of $Y(t)$ (hint: the moment generator function for a normal distribution, with expected value μ and variance σ^2 at the point a is equal to $e^{a\mu+\frac{1}{2}\sigma^2 a^2}$)

b) Prove that its covariance is equal to:

$$\mathrm{Cov}(Y(t), Y(s)) = s(1-t), s < t \in (0,1)$$

c) Is Y a Brownian motion? Justify.

27. Let $Y = \max(W^2(1) + (W(2) - W(1))^2, W^2(2))$.

a) Prove that

$$Y = W^2(2) + 2W(1)(W(1) - W(2))\mathbb{1}_{\{W(1)(W(1)-W(2))>0\}}.$$

b) Derive $\mathbb{E}[Y]$.
(recall that $\int_0^\infty x f_{N(0,1)}(x)dx = \frac{1}{\sqrt{2\pi}}$).

28. Let $X = \{X(t), t \geq 0\}$ and $Y = \{Y(t), t \geq 0\}$ be stock prices of two assets, such that:

$$dX(t) = \mu_X X(t)dt + \sigma_X X(t)dW(t)$$
$$dY(t) = \mu_Y Y(t)dt + \sigma_Y Y(t)dW(t)$$

where $W = \{W(t), t \geq 0\}$ is a Brownian motion.

a) Does there exist a function f such that $f(t, X(t)) = W(t), \forall t$?

b) Identify the process $X(t)Y(t)$.

29. Let $\{W(t), t \geq 0\}$ be a Brownian motion, and for $t \in [0,1]$, define

$$X(t) = W(t) - tW(1)$$

a) Show that $X = \{X(t), t \in [0,1]\}$ has normal distribution, and compute its covariance function.

b) For $t \geq 0$, define

$$Y(t) = (1+t)X\left(\frac{t}{1+t}\right)$$

Show that Y is a Brownian motion.

30. Let $M = \{M(t), t \geq 0\}$, with

$$M(t) = \exp\left\{-\frac{1}{2}\int_0^t b^2(s)ds - \int_0^t b(s)dW(s)\right\}.$$

Derive the corresponding stochastic differential equation. Is M an integrable process?

31. Define the process $X = \{X(t) = W(t) - tW(1), t \in [0,1]\}$. This process is known as the *Brownian bridge*.

a) Derive the quadratic variation of the Brownian bridge X.

b) Determine the mean and covariance functions of X.

c) Every other continuous Gaussian process indexed by the interval [0, 1] that has the same mean and covariance function as the Brownian bridge is also a Brownian bridge. Show that the process **Z** defined by $\{Z(t) = tW(\frac{1}{t} - 1), t \in (0,1]\}$, with $Z(0) = 0$, is a Brownian bridge.

a)

$$[X,X]_t = \lim_{n\to\infty}\sum_{i=1}^{n}\left(W(\tfrac{ti}{n}) - \tfrac{ti}{n}W(1) - W(\tfrac{t(i-1)}{n}) - \tfrac{t(i-1)}{n}W(1)\right)^2$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(W(\tfrac{ti}{n}) - W(\tfrac{t(i-1)}{n}) - \tfrac{t}{n}W(1)\right)^2$$

Let $Z_i = W(\frac{ti}{n}) - W(\frac{t(i-1)}{n}) - \frac{t}{n}W(1)$. Then it follows that Z_i is normally distributed (sum of normally distributed random variables), with $\mathbb{E}[Z_i] = 0$ and

$$\mathrm{Var}[Z_i] = \mathrm{Var}\left[W(\tfrac{ti}{n}) - W(\tfrac{t(i-1)}{n})\right] + \left(\frac{t}{n}\right)^2\mathrm{Var}[W(1)] - 2\frac{t}{n}\mathrm{Cov}\left(W(\tfrac{ti}{n}) - W(\tfrac{t(i-1)}{n}), W(1)\right)$$
$$= \frac{t}{n} + \left(\frac{t}{n}\right)^2 - 2\frac{t}{n}\left(\frac{ti}{n} - \frac{t(i-1)}{n}\right) = \frac{t}{n}\left(1 - \frac{t}{n}\right)$$

Moreover, $\{Z_i\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables. Therefore, by the strong law of large numbers:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(W(\tfrac{ti}{n}) - W(\tfrac{t(i-1)}{n}) - \frac{t}{n}W(1)\right)^2 = \lim_{n\to\infty} n\sum_{i=1}^{n}\frac{Z_i^2}{n} = \lim_{n\to\infty} n\,\frac{t}{n}\left(1 - \frac{t}{n}\right) = t.$$

b)

$$\mathbb{E}[X(t)] = \mathbb{E}[W(t) - tW(1)] = 0.$$
$$\mathrm{Cov}(X(t),X(s)) = \mathrm{Cov}(W(t) - tW(1), W(s) - sW(1))$$
$$= \mathrm{Cov}(W(t),W(s)) - s\mathrm{Cov}(W(t),W(1)) - t\mathrm{Cov}(W(s),1) + ts\mathrm{Var}[W(1)]$$
$$= \min(t,s) - st - st + ts = \min(s,t) - st.$$

c)

$$\mathbb{E}[Z(t)] = \mathbb{E}[t\,W(\tfrac{1}{t} - 1)] = 0$$
$$\mathrm{Cov}(Z(t),Z(s)) = \mathrm{Cov}(t\,W(\tfrac{1}{t} - 1), s\,W(\tfrac{1}{s} - 1)) = st\,\mathrm{Cov}(W(\tfrac{1}{t} - 1), W(\tfrac{1}{s} - 1))$$
$$= st\,(\min(1/t, 1/s) - 1) = \min(s,t) - st.$$

32. Derive the variation of order 3, defined as follows;

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(W\left(t\frac{i}{n}\right) - W\left(t\frac{i-1}{n}\right)\right)^3$$

for a Brownian motion, knowing that the moments of a normal distribution of odd order are zero.

33. Let $X = \{X(t) = tW(t), t \geq 0\}$, where $\{W(t), t \geq 0\}$ is a Brownian motion. Derive the quadratic variation $[X,X]_t$, for all t.

34. Let X be the solution of the SDE:

$$dX(t) = \alpha X(t)dt + \sigma X^{\beta}(t)dB(t), \quad X(0) = x_0$$

where α, σ and x_0 are constants.

   a) Determine the constants a, b and c so that the process $\{Y(t) = \exp\{-aX^b(t) + ct\}, t \geq 0\}$ is a martingale.
   a) Compute $\mathbb{E}[X(t)]$ when $\beta = 0.5$.

35. Let g be a function such that the Ito's integral is well defined: $I(t) = \int_0^t g(s)dW(s)$, for all $t \geq 0$. Define the continuous time stochastic process $\{M(t), t \geq 0\}$, with:

$$M(t) = I^2(t) - \int_0^t g^2(s)ds$$

Prove that $\{M(t), t \geq 0\}$ is a continuous-time martingale.

In order to prove the assertion, we use Ito's formula do derive the SDE corresponding to M and we see that its drift is zero:

$$dM(t) = dI^2(t) - g^2(t)dt = \frac{1}{2}g^2(t)dt + 2g(t)I(t)dW(t) - g^2(t)dt = 2g(t)I(t)dW(t)$$

The adaptability of M w.r.t the filtration, as well as the integrability condition, are also both consequences of the fact that M is solution of the SDE.

36. Fix $s \in [0, \infty)$ and define $Y(t) = W(t+s) - W(s)$, for $t \geq 0$. Prove that $\{Y(t), t \geq 0\}$ is also a Brownian motion with the same drift and scale parameters.

37. Prove that the Itô's integral for any process in $\mathcal{L}^2$ is an martingale.

38. Let $f : [0, T] \mapsto \mathbb{R}$ be a deterministic function, with $\int_0^t f^2(s)ds < \infty$. Prove the following:

   a) $\int_0^t f(s)dW_s$ has normal distribution, and provide the parameters.

   b) Prove that the process $\left\{\exp\left(\int_0^t f(s)dW_s - \frac{1}{2}\int_0^t f^2(s)ds\right), t \geq 0\right\}$ is a martingale.

39. By applying the generalized Itô's formula to the 2-dimensional process $\{(X_t, Y_t), t \geq 0\}$ with the function $F(x, y) = xy$, show the integration by parts formula

$$X_tY_t = X_0Y_0 + \int_0^t X_s dYs + \int_0^t Y_s dX_s + VAR[X, Y]_t$$

assuming the necessary integrability conditions which are necessary to make sense of the previous formula. In which case we get the usual formula for the derivative of the product?

40. a) Let $\{W(t), t \geq 0\}$ be a Brownian motion.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(W(\tfrac{ti}{n}) - W(\tfrac{t(i-1)}{n})\right)^2 = t$$

   for $0 < \frac{t}{n} < \ldots < \frac{ti}{n} < \ldots < \frac{tn}{n} = t$. Use this result to calculate

$$\lim_{n\to\infty} \sum_{i=1}^{n} W\left(\tfrac{ti}{n}\right) \times \left(W\left(\tfrac{ti}{n}\right) - W\left(\tfrac{t(i-1)}{n}\right)\right)$$

   (You may want to use the following relation: $2a(b-a) = (b^2 - a^2) - (b-a)^2$)

   b) Prove that any continuous stochastic process or function that has non-zero quadratic variation must have infinite total variation.

a) We use the suggestion, with $a = W(ti/n)$ and $b = W(t(i-1)/n)$. Then:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} W(\tfrac{ti}{n}) \left(W(\tfrac{ti}{n}) - W(\tfrac{t(i-1)}{n})\right) &= -\frac{1}{2}\left[\lim_{n\to\infty} \sum_{i=1}^{n} \left(W^2(\tfrac{t(i-1)}{n}) - W^2(\tfrac{ti}{n})\right)\right. \\
&\quad \left. - \lim_{n\to\infty} \sum_{i=1}^{n} \left(W(\tfrac{ti}{n}) - W(\tfrac{t(i-1)}{n})\right)^2\right] \\
&= \frac{1}{2}\left[\lim_{n\to\infty} \sum_{i=1}^{n} \left(W^2(\tfrac{ti}{n}) - W^2(\tfrac{t(i-1)}{n})\right) + t\right] \\
&= \frac{1}{2}\left[W^2(t) + t\right]
\end{aligned}$$

in view of the quadratic variation of the Brownian motion being equal to t.

b) Let $\{X(t), t \geq 0\}$ and for any t, define the partition $0 = t_0 \leq t_1 \leq \ldots \leq t_n = t$. Then

$$[X,X]_t = \lim_{n\to\infty} \sum_{k=1}^{n} (X(t_k) - X(t_{k-1}))^2 \leq \lim_{n\to\infty} \sum_{k=1}^{n} |X(t_k) - X(t_{k-1})| \times \max_i |X(t_i) - X(t_{i-1})|$$

The continuity of X implies that $\lim_{n\to\infty} \max_i |X(t_i) - X(t_{i-1})| = 0$ and thus, as the quadratic variation is positive and finite, it means that we need to have:

$$\lim_{n\to\infty} \sum_{k=1}^{n} |X(t_k) - X(t_{k-1})| = \infty$$

which proves the result.

41. Let $\{X(t) = W(t) - tW(1), t \in [0,1]\}$ be the Brownian bridge.

a) Does $\{X(t), t \in [0,1]\}$ have independent increments?

b) Let $Y(t) = X(1) - t$. Show that $\{Y(t), t \in [0,1]\}$ is again a Brownian bridge, i.e. has the same distribution as $\{X(t), t \in [0,1]\}$.

c) Calculate the conditional expectation $\mathbb{E}[X(t)|X(s)]$, for $0 \leq s < t \leq 1$.

a) For $s < t$:

$$\begin{aligned}\operatorname{Cov}(X(t) - X(s), X(s)) &= \operatorname{Cov}(W(t) - W(s) - (t-s)W(1), W(s) - sW(1)) \\ &= \operatorname{Cov}(W(t) - W(s), W(s)) - s\operatorname{Cov}(W(t) - W(s), W(1)) \\ &- (t-s)\operatorname{Cov}(W(1), W(s)) + s(t-s)\operatorname{Var}(W(1)) \\ &= 0 - s(t-s) - (t-s)s + s(t-s) = s(s-t) \neq 0\end{aligned}$$

which means that it does not have independent increments.

c) In order to solve this exercise we may use the alternative formulation of the Brownian bridge:

$$X(t) = W(t) - tW(1) = (1-t)W\left(\frac{1}{1-t}\right)$$

Then

$$\begin{aligned}\mathbb{E}[X(t)|X(s)] &= \mathbb{E}\left[(1-t)W\left(\frac{1}{1-t}\right)\middle|(1-s)W\left(\frac{1}{1-s}\right)\right] \\ &= \frac{1-t}{1-s}\mathbb{E}\left[(1-s)W\left(\frac{1}{1-t}\right)\middle|(1-s)W\left(\frac{1}{1-s}\right)\right] \\ &= \frac{1-t}{1-s}(1-s)W\left(\frac{1}{1-s}\right) = \frac{1-t}{1-s}X(s).\end{aligned}$$

42. Show, from the definition of Itô integral, that:

$$\int_0^t s dW(s) = tW(t) - \int_0^t W(s) ds.$$

By definition of Itô integral, and assuming all the necessary conditions, it follows that:

$$\begin{aligned}\int_0^t sdW(s) &= \lim_{n\to\infty}\sum_{i=1}^{n} t_{i-1}\left(W(t_i)-W(t_{i-1})\right) = \lim_{n\to\infty}\sum_{i=1}^{n} t_{i-1}W(t_i)-t_{i-1}W(t_{i-1})\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} t_{i-1}W(t_i)-t_{i-1}W(t_{i-1})+t_iW(t_i)-t_iW(t_i)\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}(t_iW(t_i)-t_{i-1}W(t_{i-1}))-(t_i-t_{i-1})W(t_i)\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}(t_iW(t_i)-t_{i-1}W(t_{i-1}))-\lim_{n\to\infty}\sum_{i=1}^{n}(t_i-t_{i-1})W(t_i)\\ &= tW(t)-\int_0^t W(s)ds\end{aligned}$$

in view of the telescopic property, the usual definition of integral, and the finiteness of all the limits, where $0=t_0\leq t_1=\frac{1}{n}\leq t_2=\frac{2}{n}\ldots\leq t_n=t$.

43. Let $b\in\mathbb{R}$ be a constant. For some point $x\in\mathbb{R}$, let:

$$X=\left\{X(t)=x+\int_0^t bX(s)ds+W(t), t\geq 0\right\}$$

a) Show that

$$dX(t)=bX(t)dt+dW(t)$$

admits one unique strong solution.

b) Show that $X(t)$ may be written as follows:

$$X(t)=e^{bt}\left[x+\int_0^t e^{-bs}dW(s)\right]$$

a) The SDE admits one unique solution if the following conditions on the drift $(\mu(t,X(t))=bX(t))$ and diffusions $(\sigma(t,X(t))=1)$ parameters hold:

- $|\mu(t,x)-\mu(t,y)|=|bx-by|=|b||x-y|$
- $|\sigma(t,x)-\sigma(t,y)|=|1-1|=0$
- $|\mu(t,x)|+|\sigma(t,x)|=|bx|+1=|b|(|x|+1)$.

b) Let $Y(t)=x+\int_0^t e^{-bs}dW(s)$. Then, applying the derivative, it follows that

$$dY(t)=e^{-bt}dW(t)$$

Now let $f(t,y)=e^{bt}y$, with $\frac{\partial f}{\partial t}=be^{bt}y$, $\frac{\partial f}{\partial y}=e^{bt}$ and $\frac{\partial^2 f}{\partial y^2}=0$. Then in view of Itô's formula, it follows that:

$$de^{bt}Y(t)=\left[be^{bt}Y(t)+e^{bt}\times 0+\frac{1}{2}e^{-bt}\times 0\right]dt+e^{bt}e^{-bt}dW(t)=be^{bt}Y(t)dt+dW(t)$$

i.e.:

$$dX(t)=bX(t)+dW(t).$$

As the SDE admits only one solution, then it means that in fact:

$$X(t)=x+\int_0^t bX(s)ds+W(t)=bX(t)+dW(t).$$

44. Let U a random variable, with uniform distribution in the interval $[0,1]$, independent of W. Define a new process $\left\{\tilde{W}(t),t\geq 0\right\}$, with:

$$\tilde{W}(t)=\begin{cases}W(t) & t\neq U\\ 0 & t=U\end{cases}$$

Check if $\left\{\tilde{W}(t), t \geq 0\right\}$ is a Brownian motion.

The vectors $(W(t_1), W(t_2), \ldots, W(t_n))$ and $(\tilde{W}(t_1), \tilde{W}(t_2), \ldots, \tilde{W}(t_n))$ are equal unless $U \in \{t_1, t_2, t_n\}$, which has probability zero (as U is a continuous random variable). Hence they have the same distributional properties. However, contrary to the process $\{W(t)\}$, $\{\tilde{W}(t)\}$ does not have continuous sample paths with probability one. In fact, $\lim_{t\to U} \tilde{W}(t) = W(U)$ but $P(W(U) = 0) = 0$. So with probability 1, $\tilde{W}$ has a discontinuity at U.

45. Let $\{W^{(1)}(t), t \geq 0\}$ and $\{W^{(2)}(t), t \geq 0\}$ be two independent Brownian motions. Consider the process $\{X(t), t \geq 0\}$, where

$$X(t) = a\left(W^{(1)}(t) + W^{(2)}(t)\right).$$

Prove that for a particular choice of a, the process is also a Brownian motion.

46. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be partition of $[0, 1]$. Compute the stochastic integral

$$X(t) = \int_0^t F(s)dW(s)$$

where F is the step function $F(s) = \sum_{i=0}^{n-1} f_i \mathbb{1}_{(s\in(t_i,t_{i+1}[)}$, with f_i's being deterministic constants. Compute its expected value, variance and covariance.

$$X(t) = \int_0^t F(s)dW(s) = \int_0^t \sum_{i=0}^{n-1} f_i \mathbb{1}_{(s\in(t_i,t_{i+1}[)} dW(s) = \sum_{i=0}^{n-1} f_i \left(W(t_{i+1}) - W(t_i)\right)$$

By the first Ito's isometry,

$$\mathbb{E}\left[X(t)\right] = 0.$$

By the second Ito's isometry,

$$\text{Var}[X(t)] = \mathbb{E}[X(t)^2] - \mathbb{E}[X(t)]^2 = \int_0^t F^2(s)ds.$$

Finally, the covariance is

$$\begin{aligned}
\text{Cov}(X(t), X(y)) &= \text{Cov}\left(\int_0^t F(s)dW(s), \int_0^y F(s)dW(s)\right) \\
&= \text{Var}\left(\int_0^y F(s)dW(s)\right) + \text{Cov}\left(\int_y^t F(s)dW(s), \int_0^y F(s)dW(s)\right) \\
&= \int_0^y F^2(s)ds + 0 \quad \text{(by independent increments, with } t > y\text{)}
\end{aligned}$$

47. a) Consider the following equations:

$$\begin{aligned}
dX(t) &= \mu_X X(t)dt + \sigma_X X(t)dW(t) \\
dY(t) &= \mu_Y Y(t)dt + \sigma_Y Y(t)dW(t)
\end{aligned}$$

Derive the differential equation for $X(t)Y(t)$. What can you say about this process, $Z = \{Z(t) = X(t)Y(t), t \geq 0\}$?

b) Suppose now that X and Y are such that

$$\begin{aligned}
dX(t) &= \mu X(t)dt - Y(t)dW(t) \\
dY(t) &= \mu Y(t)dt + X(t)dW(t)
\end{aligned}$$

What kind of process is $\left\{R(t) = X^2(t) + Y^2(t), t \geq 0\right\}$?

a) Both $\{X(t), t \geq 0\}$ as $\{Y(t), t \geq 0\}$ are geometric Brownian motion, driven by the same Brownian motion. Moreover, using the fact that

$$dX(t)Y(t) = dX(t)Y(t) + x(t)dY(t) + (dX(t)))(dY(t))$$

it follows that

$$\begin{aligned} d(X(t)Y(t)) &= (\mu_X X(t)dt + \sigma_X X(t)dW(t))\, Y(t) + X(t)\, (\mu_Y Y(t)dt + \sigma_Y Y(t)dW(t)) \\ &+ (\mu_X X(t)dt + \sigma_X X(t)dW(t))\, (\mu_Y Y(t)dt + \sigma_Y Y(t)dW(t)) \\ &= (\mu_X + \mu_Y + \sigma_X \sigma_Y) X(t)Y(t)dt + (\sigma_X + \sigma_Y) X(t)Y(t)dW(t) \end{aligned}$$

which means that the process $\{X(t)Y(t), t \geq 0\}$ is a Geometric Brownian motion, with drift $(\mu_X + \mu_Y + \sigma_X \sigma_Y)$ and volatility $(\sigma_X + \sigma_Y)$.

b) As $dX^2(t) = 2X(t)dX(t) + (dX(t))^2$, and the same for $dY^2(t)$, it follows that

$$\begin{aligned} dX^2(t) + dY^2(t) &= 2X(t)dX(t) + (dX(t))^2 + 2Y(t)dY(t) + (dY(t))^2 \\ &= 2\mu X^2(t)dt - 2X(t)Y(t)dW(t) + \mu^2 X^2(t)(dt)^2 \\ &\quad + Y^2(t)(dW(t))^2 - 2\mu X(t)Y(t)dtdW(t) \\ &\quad + 2\mu Y^2(t)dt + 2X(t)Y(t)dW(t) + \mu^2 Y^2(t)(dt)^2 \\ &\quad + X^2(t)(dW(t))^2 + 2\mu X(t)Y(t)dtdW(t) \\ &= 2\mu X^2(t)dt + Y^2(t)dt + 2\mu Y^2(t)dt + X^2(t)dt \\ &= (2\mu + 1)(X^2(t) + Y^2(t))dt \end{aligned}$$

which means that $\left\{R(t) = X^2(t) + Y^2(t), t \geq 0\right\}$ is a deterministic process, such that $R(t) = R(0)e^{(2\mu+1)t}$.

Chapter 5

Stochastic Differential Equations

1. Use a stochastic representation result in order to solve the following boundary value problem in the domain $[0,T] \times \mathbb{R}$:
$$\frac{\delta F}{\delta t} + \mu x \frac{\delta F}{\delta x} + \frac{1}{2}\sigma^2 x^2 \frac{\delta^2 F}{\delta x^2} = 0$$
with $F(T,x) = \ln(x^2)$, where μ and σ are known.

2. In this exercise, we want to solve the following boundary problem:
$$\begin{aligned} \frac{\delta}{\delta t} f(t,x) + \frac{1}{2}\frac{\delta^2}{\delta x^2} f(t,x) &= x f(t,x) \\ f(T,x) &= 1 \end{aligned} \tag{5.1}$$

   a) Compute $d(sW(s))$ and, using this result, prove that
$$\int_0^{T-t} W(s)ds = \int_0^{T-t} (T-t-s)dW(s).$$

   b) Argue that
$$\int_0^{T-t} W(s)\,ds \sim \mathcal{N}\left(0, \frac{(T-t)^3}{3}\right)$$

   c) Assume that
$$(Z|W(t)=x) = x(T-t) + \int_0^{T-t} W(s)ds$$
Solve problem (5.1).
Note: the probability generating function of a normal distribution with expected value μ and variance σ^2 at the point s is $e^{\mu s + \frac{1}{2}\sigma^2 s^2}$.

a) Using Ito's formula, it follows that
$$d(sW(s)) = W(s)ds + sdW(s)$$
Therefore, if we integrate the above formula from time 0 to time $T-t$, it follows that:
$$\begin{aligned} sW(s)|_{s=0}^{s=T-s} &= \int_0^{T-t} W(s)ds + \int_0^{T-t} sdW(s) \\ (T-t)W(T-t) = \int_0^{T-t}(T-t)dW(s) &= \int_0^{T-t} W(s)ds + \int_0^{T-t} sdW(s) \\ \int_0^{T-t} W(s)ds &= \int_0^{T-t}(T-t-s)dW(s). \end{aligned}$$

b) Ito's integral are normally distributed, so it follows that $\int_0^{T-t} W(s)ds$ is also normal distributed, in view of the last question. Its expectation and variance are the following (use Ito's isometries):

$$\mathbb{E}\left[\int_0^{T-t} W(s)ds\right] = \mathbb{E}\left[\int_0^{T-t} (T-t-s)dW(s)\right] = 0$$

$$\mathbb{E}\left[\left(\int_0^{T-t} W(s)ds\right)^2\right] = \mathbb{E}\left[\left(\int_0^{T-t} (T-t-s)dW(s)\right)^2\right] = \int_0^{T-t} (T-t-s)^2 ds = \frac{1}{3}(T-t)^3.$$

c) According to Feyman-Kac formula, it follows that the solution to problem (1) is given by:

$$f(t,x) = \mathbb{E}[e^{-\int_0^t r(X(s))ds}\Phi(X(T))|X(t)=x]$$

with, in this case, $r(x)=x$, and $dX(t)=dW(t)$, i.e., $X(T)=X(t)+(W(T)-W(t))$. Thus we need to compute:

$$\begin{aligned} f(t,x) &= \mathbb{E}[e^{-\int_t^T W(s)ds}|W(t)=x] \\ &= \mathbb{E}[e^{-x(T-t)-\int_0^{T-t} W(s)ds}] = e^{-x(T-t)} g_{N(0,\frac{1}{3}(T-t)^3)}(-1) \end{aligned}$$

where $g_Y(s)$ is the moment generating function of the r.v. Y at point s. Thus:

$$f(t,x) = e^{-x(T-t)}e^{-\frac{1}{3}(T-t)^3)}.$$

3. Solve the following boundary value problems using the Feynman-Kac formula:

   a) $\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = xu$, with $u(x,T)=1$

   b) $\frac{\partial u}{\partial t} + \alpha(\theta - x)\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0$, with $u(x,T)=F(x)$

4. Solve the following boundary value problem:

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,x) = 0$$
$$F(T,x) = x^2 - x$$

where σ is a known constant.

Recall that the solution of the PDE

$$\frac{\delta}{\delta t}f(t,x) + \mu(x)\frac{\delta}{\delta x}f(t,x) + \frac{1}{2}\sigma(x)^2\frac{\delta^2}{\delta x^2}f(t,x) = r(x)f(t,x)$$
$$f(T,x) = \Phi(x)$$

is:

$$f(t,x) = \mathbb{E}[e^{\int_t^T -r(X(s))ds}\Phi(X(T))|X(t)=x]$$

where $X=\{X(t), t\geq 0\}$ is a diffusion with drift $\mu(.)$ and volatility $\sigma(.)$.
In this particular case, the diffusion is such that:

$$dX(t) = \sigma dW(t),$$

which means that $X(t) \sim \mathcal{N}(0,\sigma^2 t)$. Therefore,

$$\begin{aligned} f(t,x) &= \mathbb{E}[e^{-r(T-t)}(X^2(T)-X(T))|X(t)=x] = e^{-r(T-t)}(\mathbb{E}[X^2(T)|X(t)=x] - \mathbb{E}[X(T)|X(t)=x]) \\ &= e^{-r(T-t)}(\mathbb{E}[(X(t)+X(T)-X(t))^2|X(t)=x] - \mathbb{E}[X(T)-X(t)+X(t)|X(t)=x]) \\ &= e^{-r(T-t)}(\mathbb{E}[(X(T)-X(t))^2] + x^2 - x) = e^{-r(T-t)}(\sigma^2(T-t)+x^2-x) \end{aligned}$$

5. Consider the following boundary value problem:

$$\frac{\partial F}{\partial t} - \frac{1}{1-t}\frac{\partial F}{\partial x} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0$$
$$F(T,x) = x^2$$

Compute the solution of such problem.

6. Solve the following boundary value problem:

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) = 0, \quad t \in [0,T], x > 0$$
$$F(T,x) = (\ln(x))^2$$

7. Solve the following differential equation:

$$\frac{\partial F(t,x)}{\partial t} + rx\frac{\partial F(t,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F(t,x)}{\partial x^2} = 0$$

for $t \in [0,T]$, such that $F(T,x) = \left[\ln(x^2) - K\right]^+$.

8. Consider the following problem:

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + k(t,x) = 0$$

for $t \leq T$, with $F(T,x) = g(x)$, and μ, σ and k being known functions. Prove that the solution of this problem is given by:

$$F(t,x) = \mathbb{E}\left[g(X(T))|X(t) = x\right] + \int_t^T \mathbb{E}\left[k(s,X(s))|X(t) = x\right] ds$$

where X is solution of the following SDE:

$$dX(s) = \mu(s,X(s))ds + \sigma(s,X(s))dW(s), \quad \text{with } X(t) = x.$$

Hint: Define $Z(s) = F(s,X(s))$ and apply Ito's formula.

From Ito's formula and the definition of F, it follows that

$$dZ(t) = -k(t,X(t))dt + \sigma(t,X(t))\frac{\partial F(t,x)}{dx}|_{x=X(t)}dW(t)$$

Therefore

$$Z(T) - Z(t) = F(T,X(T)) - F(t,X(t)) = -\int_t^T k(s,X(s))ds + \int_t^T \sigma(s,X(s))\frac{\partial F(s,x)}{dx}|_{x=X(s)}dW(s)$$

Applying the conditional expectation (given that $X(t) = x$) and the terminal condition ($F(T,x) = g(x)$), it follows that:

$$F(t,x) = \mathbb{E}\left[F(t,X(t))|X(t) = x\right] = \mathbb{E}\left[F(T,X(T))|X(t) = x\right] + \mathbb{E}\left[\int_t^T k(s,X(s))ds \,\Big|\, X(t) = x\right]$$
$$+\mathbb{E}\left[\int_t^T \sigma(s,X(s))\frac{\partial F(s,x)}{dx}|_{x=X(s)}dW(s) \,\Big|\, X(t) = x\right]$$
$$= \mathbb{E}\left[g(X(T))|X(t) = x\right] + \mathbb{E}\left[\int_t^T k(s,X(s))ds \,\Big|\, X(t) = x\right]$$

in view of the first Ito's isometry.

9. Derive the solution, F, of the following boundary problem:

$$\frac{\partial F(t,x)}{\partial t} + 2\frac{\partial^2 F(t,x)}{\partial x^2} = 0$$

with $F(T,x) = x^2$.

Consider the following SDE:

$$dX(t) = 2dW(t)$$

Then it follows that, in view of the Feynman-Kac formula:

$$F(t,x) = \mathbb{E}\left[X^2(T)|X(t) = x\right]$$

As $X(T) = 2W(T)$, then $X(t) = x$ means that $W(t) = x/2$. Then

$$\begin{aligned} F(t,x) &= \mathbb{E}\left[X^2(T)|X(t) = x\right] = \mathbb{E}\left[4W^2(T)|W(t) = 0.5x\right] = 4\mathbb{E}\left[(W(T) - W(t) + W(t))^2|W(t) = 0.5x\right] \\ &= 4\mathbb{E}\left[(W(T) - W(t))^2 + W^2(t) + 2(W(T) - W(t))W(t)|W(t) = 0.5x\right] \\ &= 4(T-t) + 4 \times 0.5^2 x^2 + 0 = 4(T-t) + x^2 \end{aligned}$$

in as consequence of the properties of the Brownian motion.

10. Consider the following boundary problem in $[0,T] \times \mathbb{R}$:

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + k(t,x) = 0$$

with $F(T,x) = f(x)$. Prove that the solution can be written as:

$$F(t,x) = \mathbb{E}\left[f(X(T))|X(t) = x\right] + \int_t^T \mathbb{E}\left[k(s,X(s))|X(t) = x\right] ds$$

with X having dynamics:

$$dX(t) = \mu(t,X(t))dt + \sigma(t,X(t))dW(t)$$

Applying Ito´s lemma to $dF(t,X(t))$, it follows that:

$$\begin{aligned} dF(t,X(t)) &= \left(\frac{\partial F}{\partial t} + \mu(t,X(t))\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 t, X(t))\frac{\partial^2 F}{\partial x^2}\right) dt + \sigma(t,X(t))\frac{\partial F}{\partial x} dW(t) \\ &= -k(t,X(t))dt + \sigma(t,X(t))\frac{\partial F}{\partial x} dW(t) \end{aligned}$$

Thus, integrating from t to T:

$$F(T,X(T)) - F(t,X(t)) = -\int_t^T k(s,X(ds))ds + \int_t^T \sigma(s,X(s))\frac{\partial F}{\partial x} dW(s).$$

Applying conditional expectation, it follows that:

$$\mathbb{E}\left[F(T,X(T))|\, X(t) = x\right] = F(t,x) - \int_t^T \mathbb{E}\left[k(s,X(s))|\, X(t) = x\right] ds + 0$$

(in view of the first Ito's isometry). Applying the terminal condition, it follows that:

$$F(t,x) = \mathbb{E}\left[f(X(T))|\, X(t) = x\right] + \int_t^T \mathbb{E}\left[k(s,X(s))|\, X(t) = x\right] ds$$

11. Solve the following differential equation:

$$\frac{\partial F(t,x)}{\partial t} + rx\frac{\partial F(t,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F(t,x)}{\partial x^2} = 0$$

for $t \in [0,T]$, such that $F(T,x) = \frac{\ln x^2 + K}{2}$.

Chapter 6

Black Scholes Model

1. Consider the usual Black-Scholes model but suppose that we introduce a new contract, whose payoff at maturity date T is equal to $S^2(T)$. Derive that the arbitrage-free pricing at time t.

   Note: $\mathbb{E}[e^{tX}] = e^{t\mu + \frac{1}{2}\sigma^2 t}, \quad X \sim \mathcal{N}(\mu, \sigma^2)$.

   Considering the Black-Scholes model, the risk-neutral valuation formula states that the arbitrage-free price, $\Pi(t, s)$, of the claim $\Phi(S(T))$ is given by:

   $$\Pi(t, s) = e^{-r(T-t)} E^Q[\Phi(S(T)|S(t) = s]$$

   where r is the rate interest (here assumed constant and continuously-compounded) and Q is the Martingale measure.
   In this case $\Phi(S(T)) = S^2(T)$, and the dynamics of S, under the Martingale measure, is $dS = rSdt + \sigma S dW$, whose solution is the Geometric Brownian Motion. Therefore

   $$E^Q[\Phi(S(T))|S(t) = s] = E^Q[S^2(T)|S(t) = s] = \int_{-\infty}^{\infty} s^2 e^{2z} f_{N((r-\frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))}(z) dz = s^2 \mathbb{E}\left[e^{2Z}\right]$$
   $$= s^2 e^{2(r-\frac{1}{2}\sigma^2)(T-t) + \sigma^2(T-t)}.$$

2. Consider an option whose payoff is equal to K if and only if the stock price of the underlying asset at the maturity date T falls in the interval $[a, b]$. Assuming the Black-Scholes model, derive the arbitrage-free pricing of this derivative.

3. Consider a standard Black-Scholes model of the form:

   $$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$
   $$dB(t) = rB(t)dt$$

   Now assume a fixed exercise date T, and define the following contingent T-claim by:

   $$X = \frac{1}{S(T)}$$

   Derive an expression for the arbitrage free price process, $\Pi(t; x)$, for this claim.

4. Consider the standard Black-Scholes model. Now assume that you write a contract regarding a digital call, with strike price K and exercise date T. This contract will give you the fixed amount of A if $S(T) \leq K$, and zero otherwise. Compute the price at time $t = 0$ of such a call.

5. Consider the standard Black-Scholes model and a option whose payoff is $\ln S(T)$ (note that if $S(T) < 1$ then this means that the holder of the option has to pay a positive amount to the writer). Determine the arbitrage-free price for this claim at any time $t < T$, assuming a constant interest rate r.

In this case, as we have that under the martingale measure Q, the stock price follows a geometric brownian motion with drift r and volatility σ, it follows that its logarithm follows the following:

$$S(T) = S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\tilde{W}(T-t)} \Leftrightarrow \ln S(T) = \ln S(t) + (r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma\tilde{W}(T-t)$$

Thus:

$$\begin{aligned}\pi(t) &= e^{-r(T-t)}E^Q[\ln S(t) + (r-\tfrac{1}{2}\sigma^2)(T-t) + \sigma\tilde{W}(T-t)] \\ &= e^{-r(T-t)}\left(\ln S(t) + (r-\tfrac{1}{2}\sigma^2)(T-t) + \sigma E^Q[\tilde{W}(T-t)]\right) \\ &= e^{-r(T-t)}\left(\ln S(t) + (r-\tfrac{1}{2}\sigma^2)(T-t)\right)\end{aligned}$$

where $\pi(t)$ denotes the price of the contract at time t.
Note: in order to have a high formalism in the resolution of this question, please note that: under the "initial" measure P, the stock price follows a geometric Brownian motion with drift r and volatility σ, it follows that its logarithm follows the following:

$$S(T) = S(t)e^{(\mu-\frac{1}{2}\sigma^2)(T-t)+\sigma W(T-t)},$$

where W is the standard Brownian motion. Then

$$\begin{aligned}S(T) &= S(t)e^{(\mu-\frac{1}{2}\sigma^2/2)(T-t)+\sigma(W(T-t)+\frac{\mu-r}{\sigma}(T-t)-\frac{\mu-r}{\sigma}(T-t))} \\ &= S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T-t)+\frac{\mu-r}{\sigma}(T-t))} \\ &= S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\tilde{W}(T-t)}\end{aligned}$$

where $\tilde{W}(t) = W(t) + \frac{\mu-r}{\sigma}t$, which, under Q and invoking the Girsanov theorem, is a standard Brownian motion.

6. Consider the standard Black-Scholes model. Derive the arbitrage free price process for the claim X, where X is given by $\frac{S(T_1)}{S(T_0)}$. The time T_0 and T_1 are given and the claim is paid out at time T_1.
7. The price of a traded security follows a geometric Brownian motion with drift 0.04 and volatility 0.2. Its current price is 40. A brokerage firm is offering, at cost 10, an investment that will pay 100 at the end of 1 year if $S(1) > (1+x)40$. Assume that the continuously compounded interest rate is 0.02. If this investment is not to give rise to an arbitrage, what is the value of x?

Use the Black-Scholes model, under which the price of a derivative whose contract function is $F(S(T))$ is given by:

$$\pi(t) = e^{-r(T-t)}E^Q[F(S(T))|S(t) = s]$$

where, in this case,

- $r = 2\%$
- $T = 1$
- $F(S(1)) = 100_{\{S(1)>(1+x)40\}}$
- Q is the measure such that under Q, the stock price follows the following dynamics:

$$dS(t) = 0.02S(t)dt + 0.2S(t)dW(t)$$

i.e., $S(t)$ is given by $S(t) = S(0)e^{(r-0.5\times0.2^2)t+0.2W(t)}$. Thus, under Q:

$$S(1) = S(0)e^{(0.02-0.5\times0.2^2)+0.2W(1)} \approx 40e^{0.2W(1)}$$

So the martingale price of this derivative is the following:

$$\pi(0) = e^{-0.02}100 \times P(40e^{0.2W(1)} > (1+x)40) = 98.0199P(W(1) > (1+x)/0.2) = 10.$$

Therefore in order to have non-arbitrage, the value x has to be such that

$$98.0199\left(1 - \Phi(\frac{1+x}{0.2})\right) = 0.10$$

Note that the Black-Scholes formula states that the price at time t of a derivative whose contract function is $F(S(T))$ is:

$$\pi(t) = e^{-r(T-t)}E^Q[F(S(T))|S(t) = s]$$

where Q is the (unique) martingale-equivalent measure.

8. Consider the standard Black-Scholes model. Derive the arbitrage free price process for the *cash-or-nothing* option, whose payoff function is

$$\Pi(S;T) = B\mathbb{1}_{\{S(T)>K\}}$$

Explain briefly the behaviour of Π as a function of T (keeping all the parameters constant) and K (keeping all the parameters constant).

Using the Black-Scholes formula, we denote by Q the risk-neutral measure of $\tilde{W}$, such that:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) = \mu S(t)dt + \sigma S_t d(\tilde{W}(t) + \tfrac{r-\mu}{\sigma}t) = rS(t)dt + \sigma S_t d\tilde{W}(t).$$

Then it follows that the price at time t of such option, when the underlying price is s, is given by:

$$\begin{aligned}
\Pi(s,t) &= e^{-r(T-t)}E^Q[B\mathbb{1}_{\{S(T)>K\}}|S(t) = s] = e^{-r(T-t)}BQ(S(T) > K|S(t) = s) \\
&= e^{-r(T-t)}BQ(S(t)e^{(r-1/2\sigma^2)(T-t)+\sigma\tilde{W}(T-t)} > K|S(t) = s) \\
&= e^{-r(T-t)}BQ\left(\tilde{W}(T-t) > \frac{\ln(K/s) - (r-1/2\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= e^{-r(T-t)}B\Phi\left(\frac{-\ln(K/s) + (r-1/2\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

Remark that Π is a decreasing function of K, as expected. The behaviour of the price as a function of time T is more complicated, probably non-monotonic. If you want to check this, plot this function.

9. Using the Black-Scholes model, compute the value at time $t = 0$ of an option with underlying stock price S and payoff:

$$g(S(T)) = (\ln(S^2(T) - K))^+$$

with strike price $K = 6$ and maturity $T = 1$.

10. Assume the Black-Scholes model. Find the arbitrage free price of the option whose payoff at maturity T is given by:

$$g(S(T)) = \max(S(T), S(0)e^{rT}) - K$$

where $K > 0$ (this option is known as *break forward*).

Assuming the Black-Scholes model, it follows that the price at time 0 is given by:

$$\pi(0;X) = e^{-rT}\mathbb{E}^Q\left[\max(S(T), S(0)e^{rT}) - K|S(0)\right]$$

where, under Q, $S(T) = S(0)e^{(r-0.5\sigma^2)T+\sigma W(T)}$. So

$$\begin{aligned}
\pi(0;X) &= e^{-rT}\mathbb{E}^Q\left[\max(S(0)e^{(r-0.5\sigma^2)T+\sigma W(T)}, S(0)e^{rT}) - K|S(0)\right] \\
&= e^{-rT}S(0)e^{rT}\mathbb{E}^Q\left[\max(e^{-0.5\sigma^2T+\sigma W(T)}, 1)|S(0)\right] - Ke^{-rT} \\
&= S(0)\mathbb{E}^Q\left[\max(e^{-0.5\sigma^2T+\sigma W(T)}, 1)|S(0)\right] - Ke^{-rT}
\end{aligned}$$

As $e^{-0.5\sigma^2T+\sigma W(T)} > 1 \Leftrightarrow -0.5\sigma^2T+\sigma W(T) > 0 \Leftrightarrow W(T) > 0.5\sigma T$, it follows that

$$\pi(0;X) = S(0)\Phi\left(0.5\sigma\sqrt{T}\right) + S(0)e^{-0.5\sigma^2T}\int_{0.5\sigma T}^{\infty} e^{\sigma x}\phi\left(\frac{x}{\sqrt{T}}\right)dx - Ke^{-rT}$$

11. Consider the following derivative X: the buyer of X obtains, at the maturity T, the value $\ln(S(T))$, where S is the stock price process. Derive its arbitrage free price and comment the result, notably in view of the possible values of the payoff of this product.

According to the Black-Scholes formula, the price of this contract, under the martingale measure Q, is given by:

$$\Pi(t;X) = e^{-r(T-t)}\mathbb{E}^Q\left[\ln(S(T))|S(t)\right]$$

where, under Q, $S(T)$ is given by:

$$S(T) = S(t)e^{(r-0.5\sigma^2)(T-t)+\sigma(W(T)-W(t))}.$$

Therefore

$$\begin{aligned}\Pi(t;X) &= e^{-r(T-t)}\mathbb{E}^Q[\ln(S(T)) + (r-0.5\sigma^2)(T-t) + \sigma\left(W(T)-W(t)\right)|S(t)] \\ &= e^{-r(T-t)}\left(\ln(S(T)) + (r-0.5\sigma^2)(T-t)\right).\end{aligned}$$

Note that in fact it can happen that the buyer of this call actually receives money, as for some values, it can happen that $\ln(S(T)) + (r-0.5\sigma^2)(T-t) < 0$, meaning that he may receive at time 0 money (instead of paying). But this is not surprising, because upon maturity, the buyer exercises for sure its option (the way we have written the contract function) and if $S(t) < 1$, when he exercises, he actually pays money to the seller of the option.

12. Consider the Black-Scholes model, with the usual assumptions. Derive the price at time $t < T$ years (where T is the maturity) of a call option with contract function:

$$g(x) = \begin{cases} 0 & x \leq K \\ 1 & x > K \end{cases}$$

where $K > 0$. Assume a continuously compounded interest rate equal to r per year. In particular, provide the price of this option at time 0 when $K = S(0)$.

Using the assumptions from the Black-Scholes model, it follows that the price at time $t < T$ is given by:

$$\pi(t;S(t)=s) = e^{-r(T-t)}\mathbb{E}^Q\left[1_{S(T)>K}|S(t)=s\right]$$

where, under the probability measure Q, S is a GBM with drift parameter equal to r:

$$S(T) = S(t)e^{(r-0.5\sigma^2)(T-t)+\sigma(W(T)-W(t))}$$

Therefore

$$\begin{aligned}\pi(t;S(t)=s) &= e^{-r(T-t)}P_Q\left(S(T) > K|S(t)=s\right) \\ &= e^{-r(T-t)}P_Q\left(se^{(r-0.5\sigma^2)(T-t)+\sigma(W(T)-W(t))} > K\right) \\ &= e^{-r(T-t)}P_Q\left((r-0.5\sigma^2)(T-t)+\sigma(W(T)-W(t)) > \ln(K/s)\right) \\ &= e^{-r(T-t)}P_Q\left((W(T)-W(t)) > \frac{\ln(K/s)-(r-0.5\sigma^2)(T-t)}{\sigma}\right) \\ &= e^{-r(T-t)}\left[1-\Phi\left(\frac{\ln(K/s)-(r-0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}})\right)\right]\end{aligned}$$

In particular, it follows that when $K = S(0)$, the price is $e^{-r(T-t)}\left[1-\Phi\left(\frac{-(r-0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}})\right)\right]$.